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[Professor / Fahmy Ibrahim Mikhail]

This is my teacher, who taught me in year 1991

The General Theory of Relativity

in the Department of Mathematics at The Faculty of Science,
in Ain Shams University.

[http://www.bibalex.org/einstein2005/part3/presentations/
Samira%20Bishay_files/frame.htm](http://www.bibalex.org/einstein2005/part3/presentations/Samira%20Bishay_files/frame.htm) —
at Faculty of Science Ain Shams University.

++ Qualifications

1- B.Sc. Special Degree in Mathematics, Cairo University, May 1941.

2- Diploma, Higher Institute of Education, Cairo, May 1942.

3- M.Sc. (Solutions of Einstein's Field Equations)
University of Wales, Cardiff, U. K. April 1949.

4- Associateship, Institute Of Education, University of London,

5- Ph.D. (Relativistic Cosmology and
Some Related Problems in General Relativity),
University of London, U. K., June 1952.

December 11, 2013 at 2:35am · Edited · Like

++ Positions Held

State Scholarship to study in England (1947-1952).

Lecturer, Applied Mathematics Department,

Faculty of Science, Ain Shams University, Cairo (1952-1956).

Associate Professor, Applied Mathematics Department,
Faculty of Science, Ain Shams University (1956-1965).

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Head of the Applied Mathematics Department,
Faculty of Science, Ain Shams University (1978-1981).

Professor Emeritus, Mathematics Department,
Faculty of Science, Ain Shams University (1981-1998).

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and
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in -

Cairo, Egypt



Of the
Academic
Year
1990 - 1991



Relativistic Mechanics :
The Theory of -
Special and General Relativity



Presented By Professor

Fahmy Ibrahim Mikhail

Head of the Mathematics Department

As written by the
Student
George F.A.M. Gabriel



This Course of Relativistic Mechanics & Theory of General Relativity

is presented by :

Prof. Dr. / Fahmy Ibrahim Michael

in 1990-1991
Faculty of Science dept. of Mathematics
Ain Shams University
Cairo - Egypt

(1)

Difficulties in Science before relativity.

1. Failure of Newton's gravitational Law (in Mechanics)

Newton's Law of (1680 - 1687)



$$F = G \frac{m_1 m_2}{r^2} \hat{r}$$

Mercury

Venus

Earth

Jupiter

Saturn

Pluto

According to this Law it was found that all planets

(9 in No) all planets rotate round the Sun in

ellipses Fixed in Space

The essence of our Subject.

Till 1845 The French astronomer Le Verrier discovered that the orbit of the planet Mercury (the nearest to the Sun) is not fixed in space.

1" = 60' minute ; 1' = 60" second

either arc Second



It rotates around the Sun by an angle 43"

Seconds of arc every Century of Mercury

i.e. 100 revolution

Relativity

* All trials to explain this difficulty to (rotation) using classical mechanics

Failed Completely

S.R 1905

1/6 the rotation

→ 7" the 1st problem

G.R 1916

43"

in Mechanics.

(2)

2. Michelson and Morley Experiment (1881 - 1887)

Huygens wave theory of light $\Rightarrow$ needs a medium to spread throughout $\Rightarrow$ invent some medium the "Ether"

$\Rightarrow$ The Ether is used to make a wave transmit $\Rightarrow$ we want to measure the speed of light



1 Time to cross the river $T_1: 2 \times \frac{100}{40} = 5 \text{ min (minutes)}$

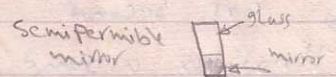
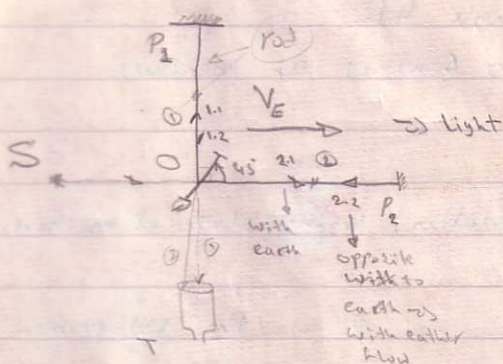
Time upstream and downstream $T_2: \frac{100}{80} + \frac{100}{20} = 1.25 + 5 = 6.25 \text{ (minutes)}$

$$T_1 < T_2$$

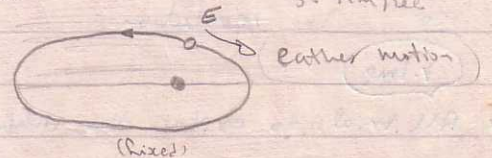
if the 30 or the 50 is unknown while given the time

$\Rightarrow$ algebraically we can get velocities

$\Rightarrow$ exactly what Michelson did.



Ether



(fixed)
motion E + Ether $\Rightarrow$ opposite flow of Ether because of E motion

Result of the experiment :

$$I_1 = I_2 \quad (\text{No Interference of light}) \quad 1881$$

3. The problem that led to S. relativity.

philosophy $\rightarrow$ Science (Past)

Scientific Philosophical ideas (1904)

Γ an idea given to Einstein as a prizegift

indeed a one who sees and aware to understand

In 1904 an international Conference for arts and Science, was held in Paris in this Conference all the difficulties in Science were discussed

Γ Henri Poincaré declared new fundamental principles to solve most of these difficulties. True Law of nature independent of the observer ~~or~~ ^{and} ~~that~~ who performs the experiment.

(i) True Laws of nature should be independent of the observer deriving them. Better stated

$$p'v'/T' = R^*$$

* Boyle's Law if T.L of nature an experimentalist in the lab. makes results $\equiv$ an observer in the metro or satellite.

(ii) This means true Laws of nature should be the same when derived by different observers having uniform relative motion.

Γ ie they have the same form

$\odot$ at rest with respect to the experiment E
 $\rightarrow$ Results R . $\odot'$ is moving w.r.to E
 gets R' Γ having the same shape

Evolution of Physics by Einstein

4

- (2) The principle of mechanics should be revised s.t
 "no velocity may exceed the velocity of light "c"
 i.e. the velocity of light should behave in mechanical problems
 as a limiting velocity. i.e. $C + v \geq C$; $C - v \geq C$

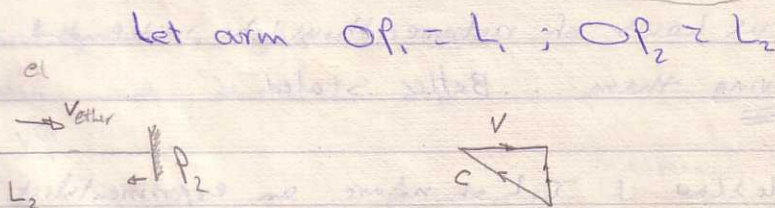
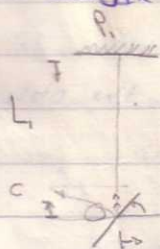


We need a new mechanics to satisfy these conditions. $C + v \geq C$
 $C - v \geq C$

4. Pre-Relativistic Solutions (for the Michelson-Morley experiment)

(i) Ether Drag Theory

(ii) Length Contraction



Let arm $OP_1 = L_1$; $OP_2 = L_2$

$$T_1 = 2 \frac{L_1}{\sqrt{C^2 - v^2}}$$

$$T_2 = \frac{L_2}{C+v} + \frac{L_2}{C-v}$$

according to the result of experiment we have $T_1 = T_2$

$$2 \frac{L_1}{\sqrt{C^2 - v^2}} = 2 \frac{L_2 C}{C^2 - v^2}$$

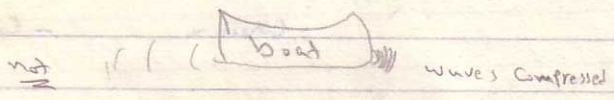
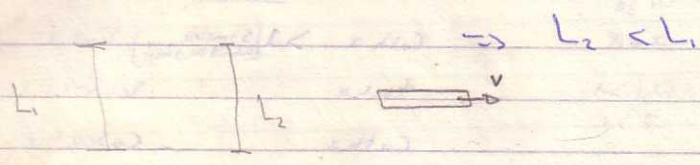
1997
Circuit Diagram

observed
= Contraction but not a physical contraction

3 meters of cloth →
washed → Contracted
(physically) in the cloth.

5

$$L_2 = \frac{L_1}{\gamma} \sqrt{c^2 - v^2} \quad \therefore L_2 = L_1 \sqrt{1 - (v/c)^2}$$



*** This is a Comparison between two readings v/mf
= Contraction but not a physical contraction

6

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

D.E'S

y	y' = dy/dx	y	dy/dx
$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
$+\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
$+\tan x$	$\sec^2 x$	$\tanh x$	$\operatorname{sech}^2 x$
$+\cot x$	$-\csc^2 x$	$\coth x$	$-\operatorname{csch}^2 x$
$+\sec x$	$\sec x \tan x$	$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$-\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

(Sep. 1991)

Show that (a) $\sinh^{-1} x = \ln(x + \sqrt{1+x^2})$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

and hence find $\frac{d}{dx} (\sinh^{-1} x)$; $\frac{d}{dx} (\tanh^{-1} x)$

$$(b) \text{ Find } \int_0^{\pi/2} \sin^4 \theta \cos^6 \theta \, d\theta ; \int_0^{\pi/2} \sqrt{\sin 2\theta} \, d\theta$$

$$(c) \text{ Solve } \frac{dy}{dx} + y \tan x = \sec x$$

Prove that $\sinh^{-1} x = \ln(x + \sqrt{1+x^2})$

$$\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 - 1})$$

$$\ln(x + \sqrt{x^2 - 1}) = \pm \ln(x + \sqrt{x^2 - 1})$$

$$* \text{ prove that } \tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad |x| < 1$$

there is two fold axes (diad axis)

On the edges (not the corners) →

2) in Cubic crystal there is (6) two fold axes



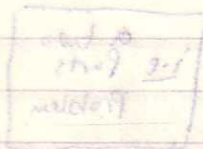
+ Reflection plane: is a mirror image to the other plane

+ I. R. A .

$\bar{1}$, $2\bar{1}2$, $3\bar{2}$, $4\bar{2}2$, $6\bar{2}$

+ Inversion Center . o image

+ Reflection planes: Images across the plane
Reflection planes along



(note)

A

Em/V

"S"

"S"

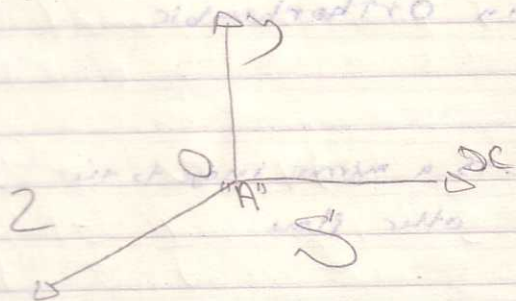
"S"

"S"

3. The Lorentz Transformation Equations:

Studying Maxwell's equations and its applications

Consider an observer "A" at rest at the origin O of a frame of reference $S(x, y, z)$



- He carries a clock to measure his time t (A is seeing a star, eating, etc.) Light a candle determined by (x, y, z, t)

- Any event E within ^{his field of consciousness} the experience of obs. "A" is completely determined by a set of 4 quantities ^{Statement 1}

$E \begin{cases} (x, y, z) & \text{to determine its position,} \\ t & \text{its time.} \end{cases}$

- Consider another obs. B at rest at the origin O' of a frame of reference $S'(x', y', z')$, he also carries a clock to measure his t' and "t"

$\rightarrow E' \begin{cases} (x', y', z') & \text{to determine its position,} \\ t' & \text{time.} \end{cases}$

Imp

+ The problem ^{then} arises ~~when~~ ~~if~~ ~~observers~~ a two parts Problem

(1) observer B is moving relative to A . O in Cairo, O' in London
no problem all at rest

Imp

* (2) The observers "A" and "B" observe the SAME event ^{no mention of time here}

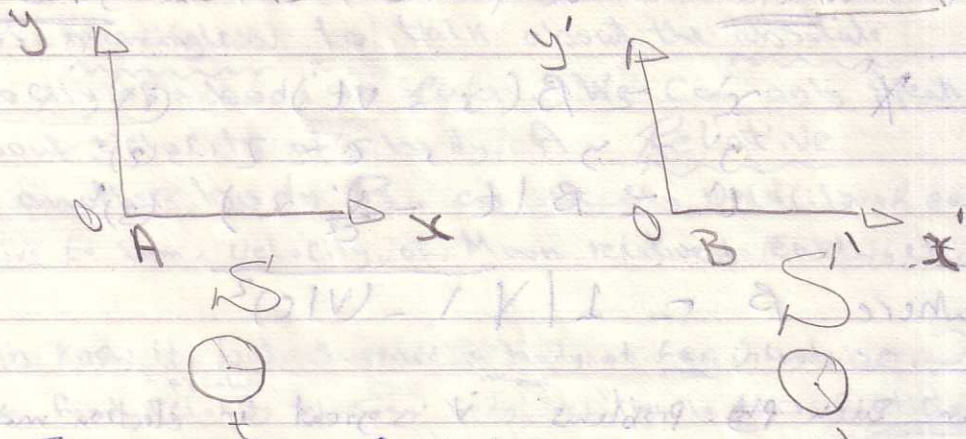
i.e. $E \equiv E'$ This is what is meant by The Same event
 the Problem is that Science must connect
 $\Rightarrow$ the two readings.

- The Problem will then be: what are the relations between 4 quantities (1) and 4 quantities (2)

* i.e. given (1) we can get (2) or vice versa.

- Answer according to Classical Mechanics
 Suppose they were completely identical in space and time. i.e. $\odot, \odot'$

i.e. A and B were in the same train



① - For Simplicity let the two obs. A, B be initially (when $t = t' = 0$) and the two frames S, S' , at this instant Coincide.

10 Let him go in any direction the way and take it
the x -axis is not a special case

(2) Then let B move along the
 x -axis with uniform relative
velocity

- Then according to classical mechanics,
we have

$$x' = x - vt \quad (1)$$

$$y' = y \quad ; \quad z' = z \quad (2)$$

$$t' = t \quad (3)$$

Lorentz wanted these eqs. to be found when
put in Maxwell eqs.

- In 1896, H.A. Lorentz rejected
these ^{(1), (2), (3)} relations, and suggested ~~was~~ instead
the following relations:

+ wanted to be proved, after he guessed it 1st

$$x' = \beta(x - vt) \quad (1)'$$

$$y' = y \quad , \quad z' = z \quad (2)'$$

$$t' = \beta(t - \frac{v}{c^2}x) \quad (3)'$$

$$\text{where } \beta = \frac{1}{\sqrt{1 - (v/c)^2}}$$

- in such phys. problems v is great in electron motion.

reduces to 0 when v is small $\rightarrow$ becomes $\beta \rightarrow 1$

in (3)' x is used to determine (t')

- Wanted is a mechanics to prove these eq^s st the physics of electrodynamics doesn't fail

- Came "Einstein" in 1905 with

Special Relativity when he was 26

+++ The Special Theory of Relativity.

Albert Einstein introduced his Special theory in 1905

(1879 - 1955)

(in a Scientific Community)

when he was only 26 years. He based this

theory on two main postulates

George (Velocity in space is a special kind of translation or transitions in the given space)

1. The 1st Postulate.

No experiment can measure, or even detect the absolute velocity of a body in space.

It is meaningless to talk about the absolute velocity of a body in space. We can only speak about velocity of a body A Relative to another body B, e.g. we say velocity of earth relative to sun, velocity of Moon relative to Earth, etc.

Newton knew it but supposed a body at far infinity st we can find absolute velocity of some body relative to this fictitious body.

True Laws of nature are independent of observers.

12

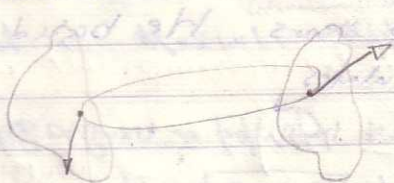
2. The 2nd Postulate.

The Velocity of Light (c) must be the same when measured by different observers having uniform relative motion. This means that the velocity of light must be independent on the velocity of observer measuring it and independent of the source emitting it.

i.e. Independent of their relative velocity.

→ This is the nature, the true nature, of light.

In double star systems rotating about each other



De Sitter
also knew it

⇒ Einstein in a sequential
deductive and inductive thought

derived Lorentz transformations

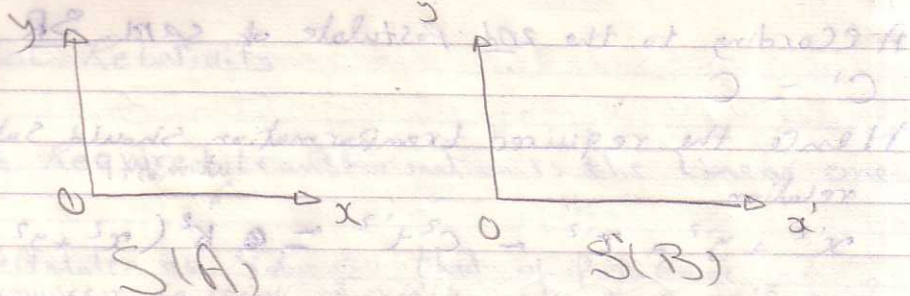
using only our two postulates

of Special relativity

We can derive the transformation eqn. suggested
by H.A. Lorentz in 1896.

— Consider (as above) the two observers A and B,
using the two frames S , S'
 $[x, y, z, t]$, $[x', y', z', t']$

Notice the sequential development of the derivation 13
 of the relations, for they are the logical prob.



1st

Consider initially S' coincide with S , i.e.

$O' \equiv O$ at this exact moment

At this instant let A (or B) exactly A and B

Send a light signal to spread in space around him.

Then let B leave A with uniform velocity V along the x -direction.

Then let each obs. observe the wave-front of this light signal

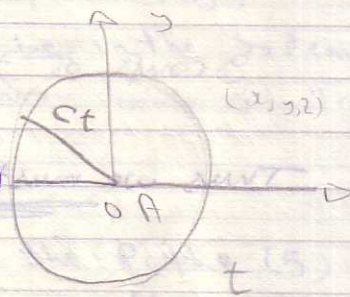
2nd According to A 's measurements:

At the instant t (according to A 's clock)

the wave front of the light signal

will be surface of the sphere

$$x^2 + y^2 + z^2 - c^2 t^2 = 0 \quad (1) \quad \text{the good Geometry}$$



- According to B 's measurements

At the instant (t') (according to B 's clock)

the wave-front of the Same light signal

will be the surface of the sphere

$$x'^2 + y'^2 + z'^2 - c'^2 t'^2 = 0 \quad (2)$$

+ wanted is a relation st given (2) we can get (1)

14

- According to the 2nd Postulate of spec. SR
 $C' = C$

2nd Hence the required transformation should satisfy the relation

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = \overset{\text{at most}}{=} K^2 (x^2 + y^2 + z^2 - c^2 t^2) \quad (3)$$

for they may use different units of measurement.

- Note that the Constant " K " is used to adjust the units used in the two systems.

Hence it can be adjusted s.t. $K = 1$ $\Rightarrow$ $\odot$

- We can also take $y' = y$; $z' = z$ $\Rightarrow$ $\odot$

$\checkmark$ Hence the required relation should satisfy the relation

$$x'^2 - c^2 t'^2 = x^2 - c^2 t^2 \quad (4)$$

- Group of transformation $\equiv$ Pure Mathematics.

4th - Thus we must prove that the transform must be linear

Special Relativity:

4th

(i) The Required transformation is the linear one.

1st Postulate of S.R. $\Rightarrow$ that of Poincaré
true laws of nature must be independent of the
observer.

- If Light ^{rays} propagate in straight lines $\Rightarrow$ if
observer moving must see it in a straight line.

+ Accepting the fact that the path of a ray of light to be
a straight line as a True Law of Light,

Accepting or not the consequence is true

then Consider a ray of light moving in the x, y plane
for simplicity. This, the path of this ray,
according to A's measurements ie in the system S
will be described by the line (From elementary geometry)

$$y = mx + d \quad (5)$$

The obs. B will describe the path (5)
of the Same ray by another
line

$$y' = m'x' + d' \quad (6)$$

This is because of true laws of nature.

(i) + The the required transformation should transform
(6) $\rightarrow$ (5) The situation is a little different

$$(ii) \quad V \rightarrow -V \quad \leftarrow \quad V \rightarrow V \quad \rightarrow \quad \frac{V}{f}$$

16. ex. transformation with $x' + x^2 + \dots$
 and by the Almighty it becomes the given one wanted.

4th
 It is well known that the only transformation that transforms a "linear expression" to another linear expression, is a linear one.

$\therefore$ The required transformation is a linear one.
 i.e. it must be of the form.

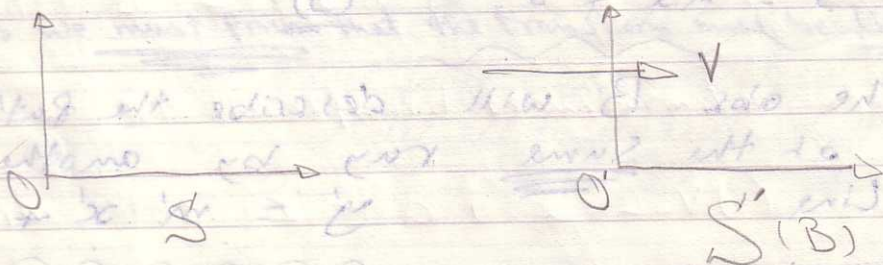
- S.t $\rightarrow x'^2 - (ct')^2 = x^2 - (ct)^2$ (4)

$x' = \beta x + \alpha t$ (7)

$t' = \gamma x + \delta t$ (8)

4 unknowns 3 eq's $\rightarrow$ we can't get the (4)
 $\rightarrow$ let us apply mechanics to help

5th
 (ii) Consider motion of O' relative to O



- But we know strictly that for O' $x' = 0$

$\therefore$ From (7) we get $\beta x + \alpha t = 0 \rightarrow$

$\therefore \frac{x}{t} = -\frac{\alpha}{\beta} = V \rightarrow \alpha = -\beta V$ (i)
 $\rightarrow$ for a more simplicity

iii) Consider motion of O relative to O'

But for O , we know that $x = 0$

$$\Rightarrow \text{from (4)} \quad \frac{x'}{t'} = \alpha t \quad \text{and} \quad t' = \delta t$$

$$\Rightarrow \frac{x'}{t'} = \frac{\alpha}{\delta} = -V \quad \text{relative velocity of train from Cairo to Alex} \Rightarrow \text{in } V$$

freely ground move from Alex to Cairo $= -V$

$$\Rightarrow \delta = -\alpha / V \quad \text{[ii]}$$

$$\Rightarrow \delta = B \quad \text{[i]}$$

Hence using [i], [ii] relations (4), (8)

will take the form

$$x' = B(x - Vt) \quad \text{[9]}$$

$$t' = Bt + \gamma x \quad \text{[10]}$$

where B, γ are two constants

16th

(iv) To Determine B, γ : Back to plane Math.

- Substituting from (9), (10) into (4) we get

hence

Experiments + trials to explain the = Science of nature.

18

$$\beta^2 (x - vt)^2 - c^2 (\beta t + \gamma x)^2 = x^2 - (ct)^2$$

* Then Comparing Coefficients to on both Sides, we get t^2

$$-c^2 = \beta^2 v^2 - c^2 \beta^2$$

$$\beta^2 (v^2 - c^2) = -c^2$$

$$\therefore \beta^2 = \frac{-c^2}{v^2 - c^2} = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - (v/c)^2}$$

$$\text{or } \beta = \frac{1}{\sqrt{1 - (v/c)^2}}$$

Coefficient of

* at $x=0$

$$-2\beta v - 2c^2 \beta \gamma = 0 \quad (\text{iii})$$

$$\gamma = -(v/c^2) \beta \quad (\text{iv})$$

^{7th}
+ Finally we get the required transformation into

the form $x' = \beta (x - vt)$

$$\beta = \frac{1}{\sqrt{1 - (v/c)^2}} \quad y' = y \quad z' = z \quad (1)$$
$$t' = \beta \left(t - \frac{v}{c^2} x \right)$$

+ which is exactly the expression Suggested

3 years by H. A Lorentz

4. Results of the Lorentz transformation Equations.

(i) The Inverse Transformation is also a Lorentz one:

- Consider the expression:

$$x' + vt' = \gamma (x - vt) + \gamma v (t - \frac{v}{c^2} x) = \gamma x (1 - \frac{v^2}{c^2}) \Rightarrow$$

$$x = \frac{x' + vt'}{\gamma (1 - \frac{v^2}{c^2})} = \gamma (x' + vt') \quad (1)'$$

- which is exactly similar to (1), except for

v is being replaced by $(-v)$ which is

expected. Similarly we can prove that.

$$t = \gamma (t' + \frac{v}{c^2} x') \quad (3)'$$

20 Perform the experiment at $S \equiv B$ and let A be the observer.

(ii) Total Interval :

Consider the differential form of the L.T

$$x' = \beta (x - vt) \quad (1) \quad dx' = \beta (dx - v dt) \quad (4)$$

$$t' = \beta \left(t - \frac{v}{c^2} x \right) \quad (2) \quad dt' = \beta \left(dt - \frac{v}{c^2} dx \right) \quad (5)$$

We can easily see that the L.T "keeps invariant the expression" $\left\{ \text{invariant} = \text{unchanged} \right\}$ in both A and B

i.e. $dx^2 + dy^2 + dz^2 - c^2 dt^2 = \text{invariant} =$

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = 1) \times \beta$$

From (4), (5) we get

$$\text{R.H.S} = dx'^2 - (c dt')^2 = \beta^2 (dx - v dt)^2 - (c \beta)^2 \left(dt - \frac{v}{c^2} dx \right)^2 =$$

$$\beta^2 dx^2 [1 - (v/c)^2] + \beta^2 dt^2 (v^2 - c^2)$$

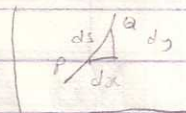
$$\text{Since } \beta = \frac{1}{\sqrt{1 - v^2/c^2}} \Rightarrow$$

$$dx^2 - c^2 \beta^2 (1 - (v/c)^2) (dt)^2 = (dx)^2 - c^2 dt^2$$

$\equiv \text{L.H.S}$

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = \text{Invariant}$$

the length between the two intervals ds



+++ This invariant expression can be taken
to represent the total interval

(in space and time [is a space interval, and a time interval])

between any two neighbouring events

$(x, y, z), t$ and $(x+dx, y+dy, z+dz), t+dt$

and is, then represented by

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2$$

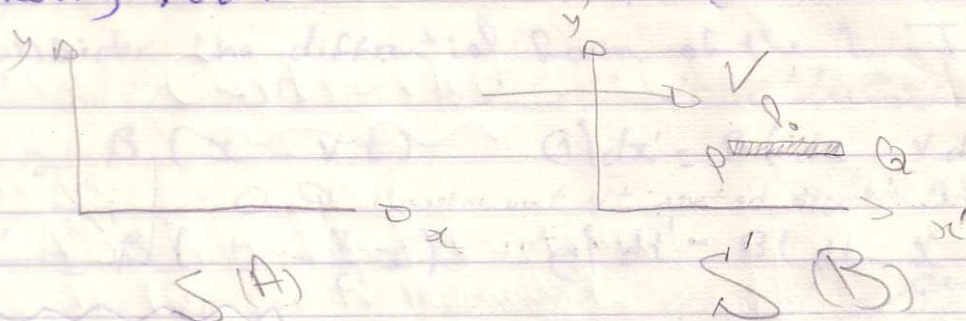
This is a physical approach, Not a Pure Mathematical
or Geometrical approach

THE SAME I MEAN THE SAME

The results here are V.Imp. from the physical point of view.

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(iii) The Fitzgerald Contraction for a moving rod.



- Let obs. B consider a rod PQ fixed parallel $//$ to the x -axis in the system S' , and let $P \equiv (x'_1, y'_1, z'_1)$ according to B 's measurements.
 $Q \equiv (x'_2, y'_2, z'_2)$ B 's measurements.

$\therefore$ Let Length of rod in B 's measurements is

$$L_B \equiv x'_2 - x'_1 \quad (1)$$

V.Imp

+++ Let Then let observer A observe the two ends of the rod PQ

AT THE SAME INSTANT t

to be at x_1, x_2 respectively

Hence The length of rod, according to A's measurements will be

$$l = x_2 - x_1 \quad (2)$$

required is a connection between (2) and (1)

But from the L.T we have

$$x'_1 = B(x_1 - vt)$$

$$x'_2 = B(x_2 - vt)$$

the key of the

problem is that the

same instant

at the same instant in S

$$x'_2 - x'_1 = B(x_2 - x_1) \Rightarrow$$

$$l = B l_0 \Rightarrow$$

$$l_0 \sqrt{1 - (v/c)^2} = l \quad (I)$$

The 1st important physical result.

$\therefore l < l_0 \Rightarrow$ when to explain this physically,

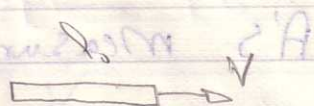
From (I) we can see that if a rod A moves along its own length, it appears

to A

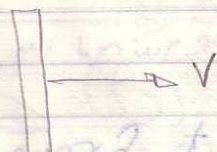
24

it APPEARS as if it was

been Contracted.



+ No Apparent Contraction
if the rod is normal to
the direction of motion.



+++ (iv) Simultaneity of Events!

If two events occur in the system S'
Simultaneously, they are not necessarily
So in the system S

- Let Observer B Carry out two events
which occur at the same instant $[t']$
according to his clock but at the
two positions x'_1, x'_2 this is possible

- Let their times, when observed by

A be t_1, t_2 respectively

(A 0)

When he saw the bird flying in the sky, he lights a candle.

(25)

From the inverse L.T, we have

$$t_1 = \beta \left(t' + \frac{v}{c^2} x'_1 \right)$$

$$t_2 = \beta \left(t' + \frac{v}{c^2} x'_2 \right)$$

$$\therefore t_2 - t_1 = \beta \frac{v}{c^2} (x'_2 - x'_1)$$

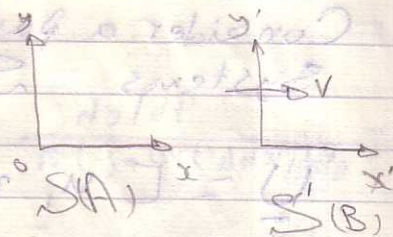
$$\therefore t_2 \neq t_1 \quad \boxed{\text{if not unless } x'_1 = x'_2}$$

+++ (V) Time Dilation of a Moving Clock:

Special Relativity:

(1) Time dilation of a moving clock:

Consider two events occurring at the same place X' in the system S' at the times



at the times t'_1, t'_2 respectively

* Let observer A observe these two events at the distant instants t_1, t_2 according to A's clock using Inverse L.T we have

$$t_1 = \gamma(t'_1 + \frac{v}{c^2}x')$$

$$t_2 = \gamma(t'_2 + \frac{v}{c^2}x')$$

$$t_2 - t_1 = \gamma(t'_2 - t'_1)$$

$$T_A = \gamma(T_B)$$

Then

$$T_A = T_B / \sqrt{1 - (v/c)^2}$$

$$\text{Then } T_A > T_B$$

∴ The time interval between the two events appear to A as if it has been elongated.

This result is expressed physically by saying: "A's moving clock appear to go slow."

ie the moving clock that is observed by A (ie appear to what is observing the moving clock)

$$T \equiv A$$

27

Composition of Velocities

Consider a particle moving w.r.t. to the two Systems S and S' . Let its velocity be

$$\underline{U} = [u, v, w] \quad \text{in } S$$

$$\underline{U}' = [u', v', w'] \quad \text{in } S'$$

$$u_i = \frac{dx_i}{dt}, \quad v_i = \frac{dy_i}{dt}, \quad w_i = \frac{dz_i}{dt}$$

$i = 1 \text{ and } 2$

$$\|\underline{U}\|^2 = u^2 + v^2 + w^2$$

$$\|\underline{U}'\|^2 = u'^2 + v'^2 + w'^2$$

* What is required is to find out the relations between these Components

$$x = \beta(x' + vt')$$

$$dx = \beta(dx' + v dt') \quad (1)$$

$$t = \beta\left(t' + \frac{v}{c^2}x'\right)$$

$$\therefore dt = \beta\left(dt' + \frac{v}{c^2}dx'\right) \quad (2)$$

$$\therefore u = \frac{dx}{dt} = \frac{\beta(dx' + v dt')}{\beta\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{dx' + v dt'}{dt' + \frac{v}{c^2}dx'}$$

* not in the strict sense of division

$$= \frac{(dx'/dt') + v}{1 + \frac{v}{c^2}(dx'/dt')} \Rightarrow$$

$$u = \frac{u' + V}{1 + (u'V/c^2)} \quad (i) \quad (2)$$

$$\text{also, } \frac{dy}{dt} = v \Rightarrow \frac{dy'}{\beta (dt' + (V/c^2)(dx'))} = \frac{dy'/dt'}{\beta (1 + (V/c^2)(dx'/dt'))}$$

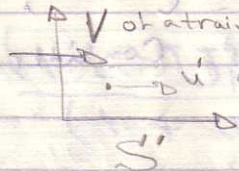
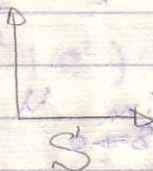
$$v = \frac{v'}{\beta (1 + (u'V/c^2))} \quad (ii); \quad w = \frac{w'}{\beta (1 + (u'V/c^2))} \quad (iii)$$

We can easily prove that $v < c$ if $v' < c$

$$u' = \frac{u - V}{1 - (uV/c^2)} \quad (i)'$$

$$v' = \frac{v}{\beta (1 - (uV/c^2))} \quad (ii)'$$

$$w' = \frac{w}{\beta (1 - (uV/c^2))} \quad (iii)'$$



Results : exam

(1) New Law of the addition of two vectors

Note that the Component u in S is the Sum of the two velocities u' & V in the Same direction, hence relation (1)

gives us a new Law for the addition of any two

This is what people of the time were longing to prove
 that the new law is correct.

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Velocities u_1, u_2 in the same direction

$$u = (u_1 + u_2) / (1 + \frac{u_1 u_2}{c^2}) \quad (III)$$

+ We can easily show that this new law (III)
 is satisfied or consistent with the second
 postulate of special relativity

1. If $u_1 = c \Rightarrow u = \frac{c + u_2}{1 + \frac{c u_2}{c^2}} = \frac{c(c + u_2)}{(c + u_2)}$

$$u = c$$

2. If $u_1 = c$ and $u_2 = c \Rightarrow$ in (III)

$$u = \frac{c + c}{1 + \frac{c^2}{c^2}} = \frac{2c}{2} = c$$

3. If $u_1 = c, u_2 = -c + \delta$

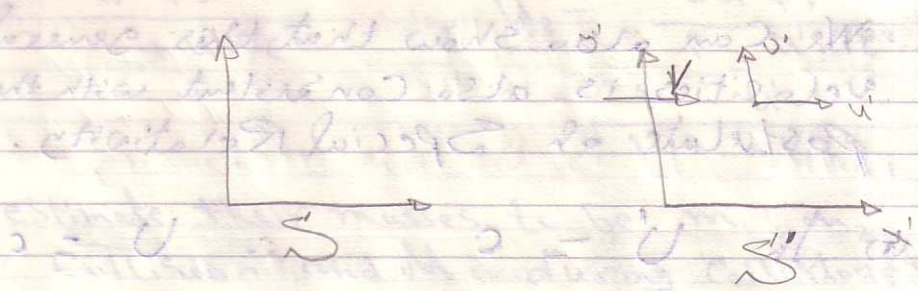
Consider

$$\lim_{\delta \rightarrow 0} u$$

Note that the comparison of the two velocities u_1 and u_2 is not valid in the same reference frame. The velocities u_1 and u_2 are measured in different reference frames.

Note that the comparison of the two velocities u_1 and u_2 is not valid in the same reference frame. The velocities u_1 and u_2 are measured in different reference frames.

Since we have two velocities u_1 and u_2 measured in different reference frames, the comparison of the two velocities is not valid.



(2) Note that Although the relative velocity of the two Systems is only in the x -direction, yet the Components in the y - and z -direction have been affected, but

if $v' = 0$ then also $v = 0$ and if $w' = 0$ then $w = 0$ i.e. no velocity in y and z direction

(3) General law of velocities

Note that the total velocity of the particle w.r to the two Systems is given by

find $U^2 = u^2 + v^2 + w^2$

Prove that $1 - (U^2/c^2) = \frac{(1 - U'^2/c^2)(1 - v^2/c^2)}{(1 + u'v/c^2)^2}$

(IV) which gives the relation between the observer and the observed.

Let us consider a particle moving with velocity u' in the x' direction in frame S' . The velocity of the particle in frame S is u . The relative velocity between the two frames is U .

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+ We Can also show that this general law of velocities is also consistent with the 2nd postulate of Special Relativity.

(ii) If $U' = c \quad \therefore U = c$

(iii) If $U' = c, V = c/A \Rightarrow U = c$

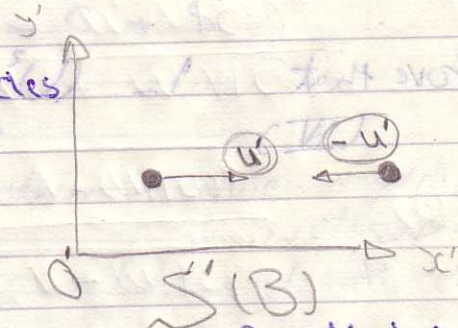
+ Variation of mass with velocity:

One of the important results of S. Relativity is the ~~change~~ ~~(different)~~ dependence of mass of a moving body.

To do so we are going to consider a simple head-on collision between two particles which are identical when

at relative rest

+ Consider two (identical) particles moving along the x' -axis in the system S' with the two velocities $u', -u'$.



Let obs. A observe these two particles to be moving with two velocities u_1, u_2 respectively

$$u_1 = \frac{u' + V}{1 + u'V/c^2}$$

$$u_2 = \frac{-u' + V}{1 - u'V/c^2} = \frac{V - u'}{1 - u'V/c^2}$$

Let A estimate their masses to be m_1, m_2 before collision and M during collision.

- All the data in this problem can be summed up into the following table:

Knowns	
	Before Collision
Velocity	u_1, u_2
mass	m_1, m_2
	During Collision
Velocity	V
mass	M
	Only
Velocity	u'_1, u'_2
mass	m'_1, m'_2

+ Applying the Conservation Laws to the data in the system

Only

1 Conservation of mass $m_1 + m_2 = M$ ①

2 Conservation of momentum $m_1 u_1 + m_2 u_2 = MV$ ②

" Can they solve the problem "

From ① and ②

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) V \Rightarrow$$

$$m_1 (u_1 - V) = m_2 (V - u_2)$$

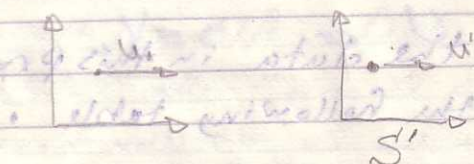
$$\therefore \frac{m_1}{m_2} = \frac{V - u_2}{u_1 - V} = \frac{V - \frac{V - u'}{1 - u'V/c^2}}{\frac{u' + V}{1 + u'V/c^2} - V} = V$$

(33)

$$\frac{m_1}{m_2} = \frac{1 + u'(v/c^2)}{1 - u'(v/c^2)} \quad (3)$$

notice B is the one measuring u' , $-u'$

From the general law of velocities IV we have



$$1 - u_1^2/c^2 = \frac{(1 - u'^2/c^2)(1 - (v/c)^2)}{(1 + u'v/c^2)^2}$$

$$1 - u_2^2/c^2 = \frac{(1 - u'^2/c^2)(1 - (v/c)^2)}{(1 - u'v/c^2)^2} \quad (5)$$

dividing $\Rightarrow$

$$\frac{1 - u_1^2/c^2}{1 - u_2^2/c^2} = \frac{(1 - u'v/c^2)^2}{(1 + u'v/c^2)^2}$$

$$\frac{1 - u'v/c^2}{1 + u'v/c^2} = \frac{\sqrt{1 - u_1^2/c^2}}{\sqrt{1 - u_2^2/c^2}} \quad (6)$$

From (3), (6) we get

$$\frac{m_1}{m_2} = \frac{\sqrt{1 - u_2^2/c^2}}{\sqrt{1 - u_1^2/c^2}} \quad (7)$$

(34)

Hence from (17) we deduce that if a particle of mass m is moving with velocity u (according to A 's measurements, then m

$$m \propto \frac{1}{\sqrt{1-(u/c)^2}} \quad \text{inverse proportionality}$$

$$\Rightarrow m = \frac{\text{Const.}}{\sqrt{1-(u/c)^2}} \quad \therefore m = \frac{m_0}{\sqrt{1-(u/c)^2}} \quad \text{---}$$

note that $m = m_0$ if $u = 0$

Thus the constant (the invariant) m_0

is called "the rest mass" of the body or the proper mass of body. "~~m~~" is

" m " is called the relative mass of the body

① $m \rightarrow 0$, $u \rightarrow c$ $\therefore m \rightarrow \frac{0}{0}$ an undetermined quantity

m will always increase but nothing is called a strictly infinite mass

Relation Between Mass and Energy

We know from the principle of work and energy that

$$dE = dW = F \cdot dr \quad (1)$$

Substituting from (2, 3) into (1) we get

$$F = \frac{d}{dt} (m u) = m \frac{du}{dt} + u \frac{dm}{dt} \quad (2)$$

$$dr = \frac{dr}{dt} \cdot dt = u \cdot dt \quad (3)$$

$$\therefore dE = \left(m \frac{du}{dt} + u \frac{dm}{dt} \right) \cdot u \cdot dt$$

$$= m u \cdot du + u u \cdot dm$$

as long as we study S.R. $\rightarrow$ acceleration $\rightarrow$

$\rightarrow$ u & du is the same direction

$$\rightarrow dE = m u du + u^2 dm \quad (4)$$

But $m = \frac{m_0}{\sqrt{1 - (u/c)^2}} = m_0 [1 - (u/c)^2]^{-1/2}$

Since $dt = \gamma dx$

$$\therefore dm = m_0 \left(-\frac{1}{2} \right) (1 - (u/c)^2)^{-3/2} \left(-\frac{2u}{c^2} du \right)$$

$$dm = \left[m_0 u / c^2 (1 - (u/c)^2)^{3/2} \right] du =$$

$$= \frac{m u du}{c^2 (1 - (u/c)^2)} = \frac{m u du}{c^2 - u^2}$$

$$\therefore m u du = (c^2 - u^2) dm \quad (5)$$

$$\therefore (dE = (c^2 - u^2) dm + u^2 dm) \text{ (the whole world works with this)}$$

$$\therefore dE = c^2 dm \quad (6)$$

(E) Kinetic or total

(36)

$$E = mc^2 + \text{Const.} \quad (8)$$

To find out the value of this Constant, we have to consider the following two cases:

[i] If E is purely kinetic i.e. (Stupid people who say this)

i.e. $E = 0$ when $u = 0$

Zero energy of body
= when body at rest
 $u = 0$

i.e. $m = m_0$

$E = 0 = mc^2$

$0 = c^2 m_0 + \text{Const.} \Rightarrow \text{Const.} = -c^2 m_0$

$E = mc^2 (m - m_0)$

[ii] If E is the total energy of body (Kinetic + Potential)

Hence we can choose our zero for measuring energy as we like. Hence taking

$E = 0$ when $m = 0$

Hence from (8) Constant = $-mc^2$

$E = mc^2$

1 $L = L_0 \sqrt{1 - (v/c)^2} \Rightarrow \frac{L}{L_0} = \frac{1}{\gamma} = \sqrt{1 - (v/c)^2}$

2 $v = \frac{u_1 + u_2}{1 + (u_1 u_2 / c^2)}$

3 $m = \frac{m_0}{\sqrt{1 - (v/c)^2}}$ 4 $E = mc^2$

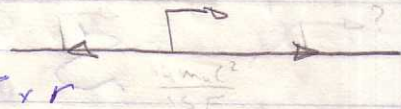
(37)

Ex. 1. A Particle started to move along a straight line under the action of a constant force resisting Force F . If its initial velocity is $\frac{12}{13}c$, find its velocity when it covers a distance $(14 m \cdot c^2 / 15 F)$. If the force reverses its direction find the distance it covers before it commencing to rest.

↑ Such a kind of problems is solved by the principle of work and energy

$$\Delta E = \text{Work done} \quad \text{resistive force}$$

i.e. $E_2 - E_1 = -F \cdot r$



$$m_2 c^2 - m_1 c^2 = \frac{14 m_0 c^2}{15}$$

$$\frac{m_0 c^2}{\sqrt{1 - (u_2/c)^2}} - \frac{m_0 c^2}{\sqrt{1 - \frac{144}{169}}} = \frac{14 m_0 c^2}{15}$$

$$\frac{m_0 c^2}{\sqrt{1 - (u_2/c)^2}} = \frac{13 m_0 c^2}{5} \quad \text{---} \quad \frac{14 m_0 c^2}{15}$$

$$\frac{1}{\sqrt{1 - (u_2/c)^2}} = \frac{14}{15} + \frac{13}{5} = \frac{25}{5} = 5 \quad \left[1 - (u_2/c)^2 \right]^{-1/2} = \frac{5}{1}$$

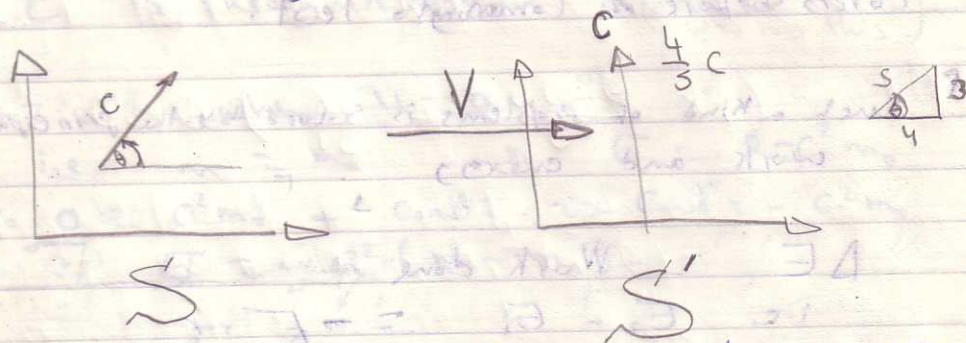
$$\left[1 - (u_2/c)^2 \right]^{1/2} = \frac{1}{5} \quad \Rightarrow \quad \frac{5}{5} = 1$$

$$1 - \left(\frac{u_2}{c} \right)^2 = \left(\frac{1}{5} \right)^2 = \left(\frac{1}{5} \right)^2 = \frac{1}{25}$$

(38)

Ex. 2) A ray of light is moving in the system S

In a direction inclined at an angle $\tan^{-1} \frac{3}{4}$ to the x -axis. Prove that it will appear in S' to be moving parallel to the y' -axis, knowing that $V = \frac{4}{5}c$ (relative velocity)



$$\text{In } S \begin{cases} u = c \cos \theta = \frac{4}{5}c \\ v = c \sin \theta = \frac{3}{5}c \\ w = 0 \end{cases} \quad \begin{cases} u' = u - V / (1 - \frac{uV}{c^2}) \\ v' = v / \gamma (1 - \frac{uV}{c^2}) \\ w' = 0 \end{cases}$$

$$u' = \frac{u - V}{1 - \frac{uV}{c^2}} = \frac{\frac{4}{5}c - \frac{4}{5}c}{1 - \frac{\frac{4}{5}c \cdot \frac{4}{5}c}{c^2}} = 0$$

$$v' = \frac{\frac{3}{5}c}{\frac{5}{3} (1 - \frac{16}{25})} = \frac{(\frac{3}{5})c}{\frac{5}{3} \times \frac{9}{25}} = c$$

$$w' = 0 = 0$$

(Prove $\rightarrow$ E.P.W.)

Hence the other form of

S, R

Geometrical Representation of Special Relativity.

1. Minkowski's 4-dimensional Continuum

From ~~PL~~ Transformation we have $\cos \theta = \gamma$ even function

$$\beta = \gamma v$$

$$x' = \beta (x - vt)$$

$$(1) \theta = -\pi \rightarrow -\frac{\pi}{2} \rightarrow -\pi/4 \rightarrow 0 \rightarrow \pi/4 \rightarrow \pi/2 \rightarrow \pi$$

$$y' = y$$

$$\beta = -1 \rightarrow 0 \rightarrow \frac{1}{\sqrt{2}} \rightarrow 1 \rightarrow \frac{1}{\sqrt{2}} \rightarrow 0 \rightarrow 1$$

$$(2) v = -ic \rightarrow 0 \rightarrow \frac{1}{\sqrt{2}} ic \rightarrow ic \rightarrow \frac{1}{\sqrt{2}} ic \rightarrow 0 \rightarrow ic$$

$$t' = \beta (t - \frac{v}{c^2} x)$$

$$(3) t'$$

$$\beta = 1 / \sqrt{1 - (v/c)^2} \quad (4)$$

- We are going to introduce the new variables defined by

$$x_1 \equiv x$$

$$x_2 \equiv y$$

$$(5)$$

$$x_3 \equiv z$$

$$x_4 \equiv ict$$

$$(i v = c \tan \theta)$$

$$\text{and } v = -ic \tan \theta$$

$$i = \sqrt{-1}$$

$$\tan \theta = \frac{v}{-ic} = \frac{vi}{-ic} = \frac{iv}{c}$$

$$\text{i.e. } \tan \theta = \frac{iv}{c}$$

$$(6)$$

From (4) we get $\beta = 1 / \sqrt{1 + \tan^2 \theta} = \frac{1}{\sec \theta}$

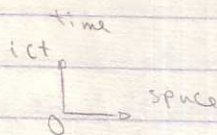
$$\text{When } \theta = 0 \rightarrow \beta = 1 \text{ and } v = 0, x' = x$$

$$\beta = \cos \theta$$

$$\theta = \frac{\pi}{4} \rightarrow \beta = 1/\sqrt{2} \text{ and } v = -ic, x' = \frac{1}{\sqrt{2}}(x - ict)$$

Using this variable ~~PLT~~ becomes $\theta = \frac{1}{\sqrt{2}}(x_1 - x_4)$

$$\text{When } \theta = \pi/2 \rightarrow \beta = 0, v =$$



(40)

$$x'_1 = \cos \theta \left(x_1 + ic \tan \theta \left(\frac{x_4}{ic} \right) \right)$$

$$\therefore x'_1 = \cos \theta (x_1 + x_4 \tan \theta)$$

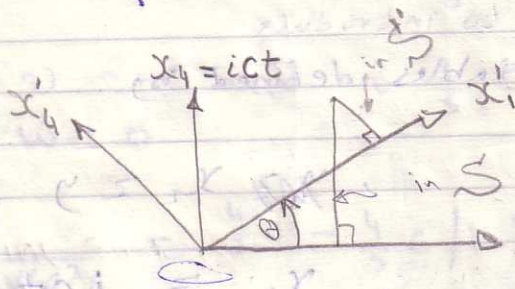
$$x'_1 = x_1 \cos \theta + x_4 \sin \theta \quad (1')$$

From L.T (3) we get $V = c \tan \theta$

$$\frac{x'_4}{ic} = \cos \theta \left(\frac{x_4}{ic} + \frac{ic \tan \theta}{c^2} x_1 \right)$$

$$\therefore x'_4 = \cos \theta (x_4 - x_1 \tan \theta)$$

$$x'_4 = x_4 \cos \theta - x_1 \sin \theta \quad (3')$$



(1'), (3') Shows That the Lorentz. T

Is Simply equivalent to a Simple

Plane Rotation of the two axes

Ox_1 , Ox_4 through an angle

θ given by $\theta = \tan^{-1} \left(\frac{iv}{c} \right)$

This idea was 1st given by Minkowski in 1909. It leads to, any event E can be represented by a point in 4-dimensional space by

(x_1, x_2, x_3, x_4) defined by above.

$$E \equiv (x_1, x_2, x_3, x_4)$$

and its coordinates in the system S' will be given by $(x'_1, x'_2, x'_3, x'_4) \equiv E'$

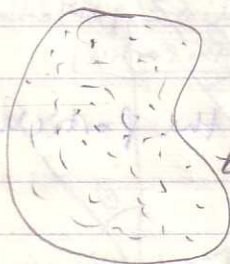
With x'_1, x'_4 are related to x_1, x_4 by (1), (3) above

$$x'_1 = x_1, \quad x'_4 = x_4$$

What is the meaning of a 4-Dim space

Physically Problem 1st we want to get Position P the point in desire for the (each person to reach)

here this style is gotten by something like telephone numbers, Codes Codes



Continuum

not today any space
electrons and every person
has his own given Symbol

every point is unique in this

Continuum

$\Rightarrow$ When we say 4-Dimensions, this doesn't mean that we live in a

Compound world, all is Codes

(42)

2. Representation of Events in Minkowski's

Space time Continuum. (Let for now call it Space)

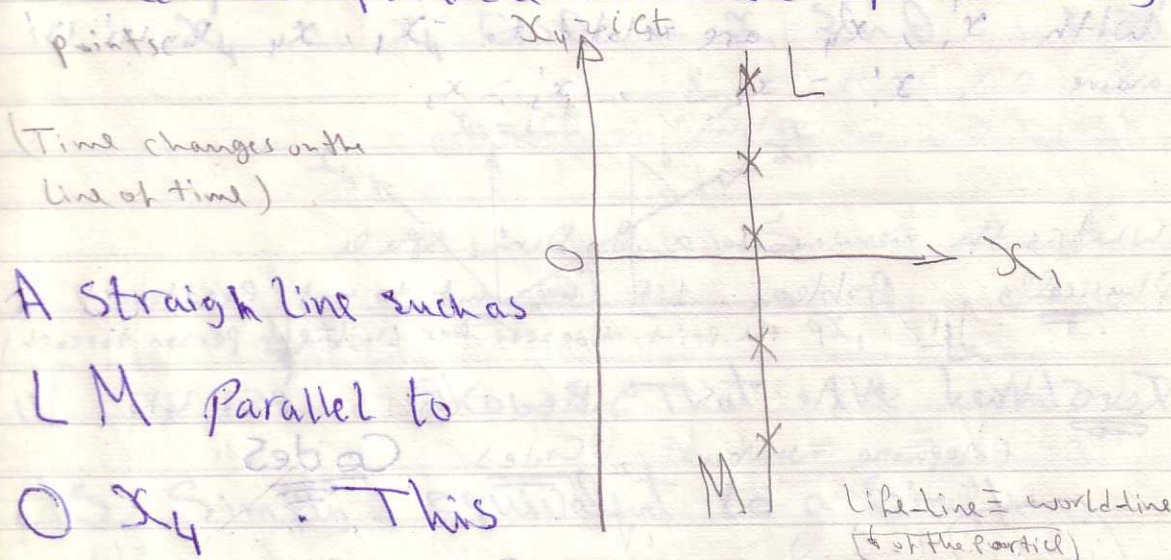
(i) A Point at Rest:

Physics: A particle at rest in the system

"S"

Geometry: In Mink.'s 4-dimensional Space

Such a particle will be represented by:



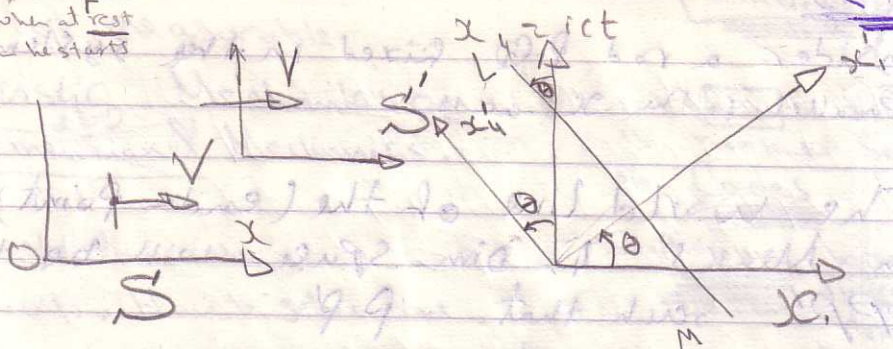
Line LM is called the world-line of the particle.

Ex. (ii) A moving particle

Consider a particle moving in S with a uniform velocity V // to the x -axis

(43)

what if x_1, x_2 is considered
 raised to space which were the
 coordinates of A when at rest
 w.r to space when he starts
 motion w.r
 to space
 he is in S'
 his twin
 otherwise



Physics:

$$x = x_0 + vt$$

$$y = \text{const.}; z = \text{const.}$$

Transformed in Mink's new representation

Geometry: $x_1 = x_0 - x_2 \cdot \tan \theta \times \frac{x_2}{c/c}$

$$x_1 = x_0 - (c \tan \theta \times x_2 / c)$$

$$\therefore x_1 = x_0 - x_2 \tan \theta$$

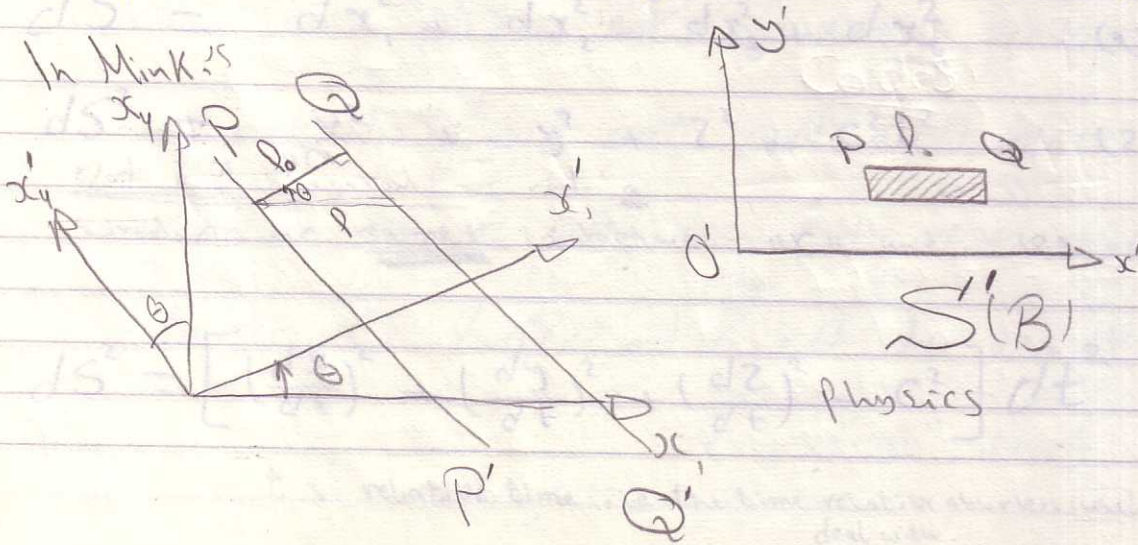
10

This relation Represents in Mink's Space,

a straight line (like-line of the particle)
 (world-line of it)

Straight line $\parallel$ to Ox_1'

Applications (1) Fitzgerald Contraction:



(44)

Consider a rod PQ fixed in the system S
Parallel to the x axis

The world line of the (end-point) P
in Mink's 4-Dim. Space will be the line
 PP' such that PP'

$$PP' \parallel OX'_4$$

Similarly the world line of the end point Q
is QQ' parallel to OX'_4

Measurable Coordinates $\equiv$ Euclidean Coordinates

here not measurable

Sensible

(45)

+ Relativistic Mechanics or 4-dimensional Mechanics.

how to draw the 4th Dim.
is nonsense because
of Codes

Continuum; how to draw
Codes
Nonsense

Here the old time when we
wanted velocity, How to get a time
here

1. The Proper Time:

Consider two neighbouring points in Mink's
4-Dimensional Space represented by
 (x_1, x_2, x_3, x_4) ; $(x_1+dx_1, x_2+dx_2,$

$x_3+dx_3, x_4+dx_4)$

The Interval between these two points

is given by

i.e is Strictly and
Completely Correct

Not any Caution as before

$$dS^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \quad (1)$$

$$dS^2 = x^2 + y^2 + z^2 - c^2 t^2 \quad (2)$$

Not a Pheathagoras or not a

Similarity of names i.e between "x" and "t"

$$dS^2 = \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 - c^2 \right] dt^2$$

$t \equiv$ relative time, i.e the time relative observers use,
deal with.

(46)

$$\therefore ds^2 = (v^2 - c^2) dt^2$$

Note ds is an imaginary since $v < c$

(for the sake of physics, we say)

For this reason, we introduce the new real invariant " τ ".

defined by

$$\boxed{c \, d\tau = i \, ds} \quad (4)$$

(2) Inside only.

(4) On the surface.

From (4), (3)

$$-c^2 d\tau^2 = (v^2 - c^2) dt^2$$

$$\therefore d\tau^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2$$

$$\therefore d\tau = \alpha \, dt$$

$$dt = d\tau / \alpha$$

where $\alpha = \sqrt{1 - (v/c)^2}$

which appears to be the time we will always use

+ note that the dimensions of the new

invariant (τ) are the same as

those of the Classical Coordinate time " τ "

In fact " τ " is identical when
 $v = 0$

* The new invariant " τ " is called
 "the proper time".

(iii) Velocity 4-Vector:

The 4-Dimensional Vector ^{geometrically} representing
 the velocity of a moving particle is

defined by:

$$\underline{v} = dx/d\tau$$

Its Components are given by

$$v^i = \frac{dx_i}{d\tau} \quad i = 1, 2, 3, 4$$

$$\therefore \underline{v} = \left[\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}, \frac{dx_4}{d\tau} \right]$$

This is geometry or geometrization of
 physics

To move to physics. $\rightarrow$

(48)

$$|d\tau = \alpha dt|$$

$$\underline{v} = \frac{1}{\alpha} \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic \frac{dt}{dt} \right]$$

$$\underline{v} = \frac{1}{\alpha} [\underline{v}, ic] \quad (II)$$

where $\underline{v} \equiv (\dot{x}, \dot{y}, \dot{z})$ is the

classical 3-dimensional velocity

+ From II we can easily see that

(magnitude) of this 4-vector is

Constant i.e. $\sum_{i=1}^4 (v^i)^2 =$

$$\frac{1}{\alpha^2} [x^2 + y^2 + z^2 - c^2] = 0$$

$$\frac{1}{1 - (v/c)^2} (v^2 - c^2) = 0$$

$$\frac{c^2}{c^2 - v^2} (v^2 - c^2) = 0$$

$$\sum_{i=1}^4 (v^i)^2 = -c^2 \quad III$$

= Constant

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}}$$

(ii) Momentum 4-Vector: (49)

The 4-dimensional vector representing the momentum of a moving particle is given by

$$\underline{p} = m_0 \underline{v} \quad \text{its components will be}$$

$$p^i = m_0 v^i \quad \text{IV}$$

$i = 1, 2, 3, 4$

$$\therefore \underline{p} = \frac{m_0}{\alpha} [\underline{v}, ic]$$

$$\text{Since } m = \frac{m_0}{\sqrt{1 - (v/c)^2}}$$

$$\underline{p} = m [\underline{v}, ic]$$

$$\therefore \underline{p} = [\underline{p}, imc] \quad \text{V}$$

We can also see from IV that

$$\sum_i^4 (p^i)^2 = m_0^2 \sum_i^4 (v^i)^2 = -m_0^2 c^2 \quad \text{VI}$$

Constant

and hence its energy is usually

(50)

From I, II we get

$$P_x^2 + P_y^2 + P_z^2 - m^2 c^2 = -m_0^2 c^2$$

$$P^2 - m^2 c^2 = -m_0^2 c^2$$

but $E = mc^2$ so $m = E/c^2$

$$P^2 - \frac{E^2}{c^4} \cdot c^2 = -m_0^2 c^2$$

$$E^2 = c^2 (P^2 + m_0^2 c^2)$$

V.Imp for physics ; but we will not use it here in problems.

$$\rightarrow \underline{U} \equiv \frac{1}{\alpha} (\underline{U}, i c)$$

$$P \equiv (\underline{P}, i m c) \equiv \left(\frac{m_0}{\alpha} \underline{U}, i \frac{m_0}{\alpha} c \right)$$

$$\sum_{i=1}^4 (P^i)^2 = -m_0^2 c^2$$

Particles of Zero Rest-Mass:

We Can easily see that only two
any Particle with the Velocity of light c [ie a photon or neutrino]

has a Zero Rest Mass, Since we know that

$$p^2 - m^2 c^2 = -m_0^2 c^2$$

(For any Particle)

But for such particles

$$p = mc$$

$$\therefore m^2 c^2 - m^2 c^2 = -m_0^2 c^2$$

$$\therefore 0 = -m_0^2 c^2$$

$$\therefore m_0 = 0$$

- But these particles are always

associated with a wave,
and hence its energy is usually

(52)

given by $h\nu = E$

$$\lambda\nu = c$$

inside
outside
on & outside

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2}$$

hence the momentum 4-Vector
of these particles (Photons & neutrinos)

Can be expressed into the form:

$$P \equiv [\underline{P}, imc] = [mc, imc]$$

3-Dim Vector

4-Dim
vector

$$P \equiv \left[\frac{h\nu}{c}, i \frac{h\nu}{c} \right]$$

Photons and neutrinos

notice
 $\| \underline{P} \| = \frac{h\nu}{c}$
V.V-imp

Note that for these particle

$$\sum_{i=1}^4 (p^i)^2 = -m_0^2 c^2 = 0$$

But these particles are always

moving with a wave

and hence its energy

The Inner product of ~~any~~
two 4-vectors :

Consider the two momentum 4-vectors

$$\underline{P} = [p, im_1c] ; \underline{R} = [r, im_2c]$$

The inner product of these two
4-vectors is defined by :

$$(\underline{P}, \underline{R}) = \sum_{i=1}^4 (P^i R^i)$$

has no physical interpretation but

Algebraically always solve problems.

$$= \underline{P} \cdot \underline{R} - m_1 m_2 c^2 =$$

$$PR \cos \theta - m_1 m_2 c^2$$

which is always used for we don't always have
the components of $\underline{P}$ and $\underline{R}$

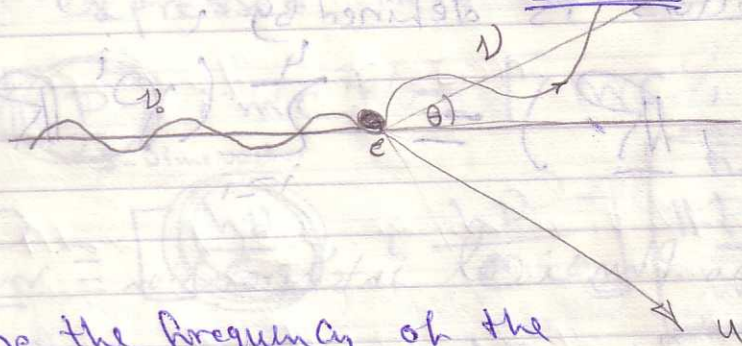
(54)

$$(P, P) = \sum_{i=1}^4 (P^i)^2 = -m_0^2 c^2$$

example (1) The Compton Effect:

Why some material scatter light and others not Silver
 $\Rightarrow$ Polishing creates free electrons.

A Collision between a photon and
a free electron at Rest



Let ν_0 be the frequency of the
incident photon ; ν of the scattered photon
 u = velocity of electron after collision.

Solution

{Initial Conditions} Photon

$$\begin{cases} \underline{P}_0 = \left[\frac{h\nu_0}{c}, i \frac{h\nu_0}{c} \right] \text{ incident} \\ \underline{P} = \left[\frac{h\nu}{c}, i \frac{h\nu}{c} \right] \text{ scattered} \end{cases}$$

(the same technique for more than one particle)
(for problem solving)

(55)

Electron $\left\{ \begin{array}{l} \underline{R}_0 = [0, i m_0 c] \text{ (before collision)} \\ \underline{R} = [\frac{m_0 u}{\alpha}, i (m_0 c / \alpha)] \text{ (after collision)} \end{array} \right.$ (bad not enough information)

1st form the momentum 4-vectors of the problem.

$$\underline{P}_0 + \underline{R}_0 = \underline{P} + \underline{R}$$

or 4 equations

Laws of Conservation.

Therefore $\underline{R} = \underline{P}_0 + \underline{R}_0 - \underline{P}$

then consider the inner product of each side with itself.

$$\begin{aligned} \underline{R}^2 &= \underline{P}_0^2 + \underline{R}_0^2 + \underline{P}^2 + 2(\underline{P}_0, \underline{R}_0) \\ &\quad - 2(\underline{P}_0, \underline{P}) - 2(\underline{R}_0, \underline{P}) \end{aligned}$$

Multiplying using the inner product of each side by itself, we get,

$$\begin{aligned} -m_0^2 c^2 &= 0 - m_0^2 c^2 + 0 + \\ &\quad 2(\underline{P}_0, \underline{R}_0) - 2(\underline{P}_0, \underline{P}) - 2(\underline{R}_0, \underline{P}) \end{aligned}$$

$$\begin{aligned} \therefore (\underline{P}_0, \underline{R}_0) &= (\underline{P}_0, \underline{P}) + (\underline{R}_0, \underline{P}) \\ &= 0 - h\nu_0 m_0 = \left[\frac{h^2 \nu^2}{c^2} \cos \theta - \frac{h^2 \nu_0^2}{c^2} \right] \\ &\quad + 0 - h\nu m_0 \Rightarrow \end{aligned}$$

(56)

$$m_0 h (\nu - \nu_0) = \frac{h \nu \nu_0}{c^2} (\cos \theta - 1) \Rightarrow$$

dividing the two sides by $\nu - \nu_0$

$$\left(\frac{c}{\nu} - \frac{c}{\nu_0} \right) = \frac{h}{m_0 c} (1 - \cos \theta) \Rightarrow$$

$$\lambda - \lambda_0 = \frac{2h}{m_0 c} \sin^2 \frac{\theta}{2}$$

where λ_0 = wave length of incident light.

λ = scattered

example (2)

Emission of Photon.

(atom emits photon)

atom excited $\rightarrow$ energy emitted $\rightarrow$ Collision of photons

An excited Atom at rest in an inertial frame, emits a photon with frequency ν , and recoils with velocity u . If M , m are the rest masses of the atom before and after the emission of the photon, Prove that the energy of the emitted photon is given by

$$h\nu = \Delta E \left(1 - \frac{\Delta E}{2Mc^2} \right)$$

$$\text{where } \Delta E = (M - m) c^2$$

S.R.

Hence a length of rod according to B's measurements is given by

$$l_0 = PQ \quad \text{where } PQ \parallel X'$$

$$l_0 = PR \quad PR \parallel X_1$$

From trigonometry, in triangle PQR

$$PQ = PR \cos \theta \Rightarrow l_0 = l \cos \theta = l \sqrt{1 - (v/c)^2}$$

$$l = l_0 \sqrt{1 - (v/c)^2} \quad \text{I}$$

$$PQ = QT$$

$$PR = AT$$

$$l = l_0 \sqrt{1 - (v/c)^2} \quad \text{II}$$

$$= l_0 \left(\frac{1 - \frac{v^2}{c^2}}{2H} \right) = l_0 \left(\frac{2H - \frac{v^2}{c^2}}{2H} \right)$$

(58)

$$PQ = PR \cos \theta$$

$$\beta = \frac{1}{\sqrt{1 - (v/c)^2}} ; \beta = \frac{1}{\sqrt{1 - (v/c)^2}}$$

$$\therefore \cancel{PQ} = \cancel{P} = \beta \sqrt{1 - (v/c)^2} \quad \text{I}$$

hence the rod appears to have been contracted.

(ii) Time Dilatation

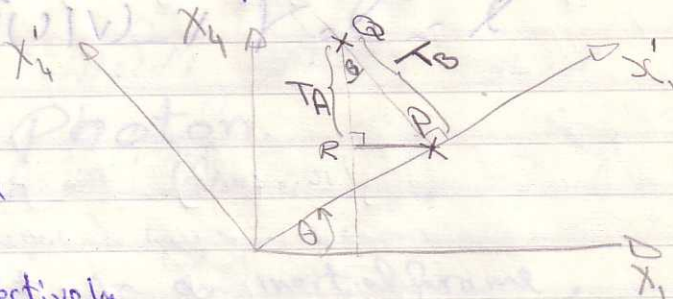
Phy. Consider two events occurring in the system S' at the same place

Such a pair of events will be represented in

Mink's 4-dimensional space by the two points

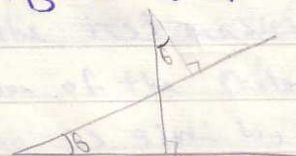
P, Q where PQ

$$\cancel{PQ} \parallel OX'_4$$

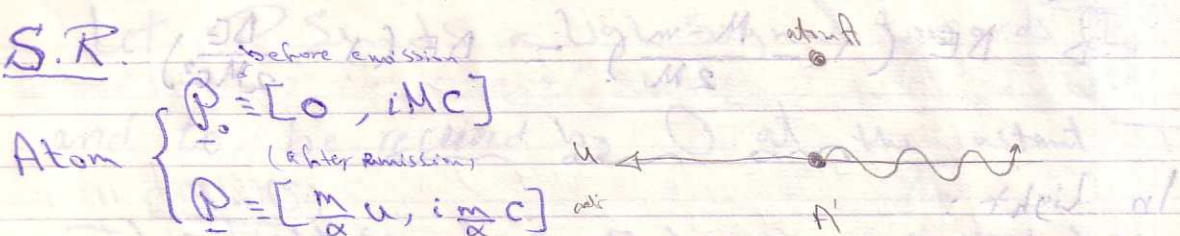


Hence the time interval between these two events according to A, B respectively are given by $T_A = QR$, $T_B = QP$

Therefore, from ΔPQR we have $T_A = T_B \cos \theta =$



$$\beta T_B \Rightarrow T_A = T_B / \sqrt{1 - (v/c)^2} \quad \text{II}$$

S.R.

Photon: $\begin{cases} \underline{K} = [\frac{h\nu}{c}, i\frac{h\nu}{c}] \\ \underline{K}_0 = [0, iMc] \end{cases}$

according to Conservation Laws

$$\underline{P} = \underline{P}_0 - \underline{K}$$

$$\Rightarrow \underline{P} = \underline{P}_0 - \underline{K}$$

Taking the inner product of the ~~two~~ each side with itself we get.

$$\underline{P}^2 = \underline{P}_0^2 + \underline{K}^2 - 2(\underline{P}_0, \underline{K}) \Rightarrow$$

$$- \left(\frac{m}{\alpha} u \right)^2 - \left(\frac{mc}{\alpha} \right)^2$$

$$- m^2 c^2 = - M^2 c^2 - 0 + 2 \frac{M h \nu}{\alpha} \Rightarrow$$

$$- m^2 c^2 = - M^2 c^2 + 2 M h \nu \Rightarrow$$

$$h \nu = \frac{(M^2 - m^2) c^2}{2 M} = \frac{(M - m)(M + m) c^2}{2 M}$$

$$= \Delta E \left(\frac{M + m}{2 M} \right) \approx \Delta E \left(\frac{2 M - M + m}{2 M} \right)$$

(60)

$$\Rightarrow DE \left(1 - \frac{M-m}{2M}\right) = DE \left(1 - \frac{DE}{2Mc^2}\right) \quad \text{2.2}$$

In light:

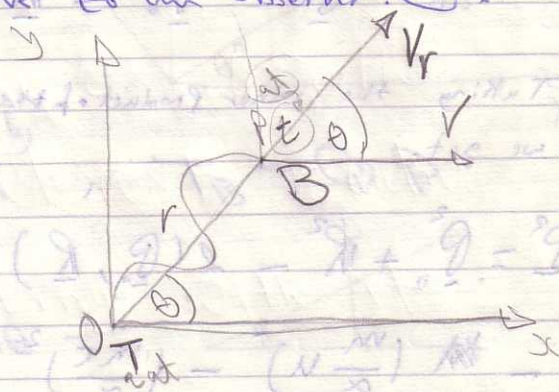
The Doppler Effect:

Using the ideas of Special Relativity we can derive a new formula for the Doppler Effect,

in the following manner.

+ Consider a source of light P which is moving with velocity V relative to an observer O .

Hence at any instant t , according to O 's clock the position of P is given by



$$x = x_0 + Vt$$

$$y = \text{Const.}, \quad z = \text{Const.}$$

$$\Rightarrow dx = V dt \quad ; \quad dy = 0, \quad dz = 0$$

$$\text{If } OP = r \quad \Rightarrow$$

$$r^2 = x^2 + y^2 + z^2 \quad \Rightarrow$$

$$2r dr = 2x dx + 2y dy + 2z dz$$

$$r dr = x dx$$

all this is algebra.

Let P sends a light signal towards O ,
and to be received by O at the instant T .

→ This light signal must have been emitted by P at an earlier instant t , such that

$$r = (T - t) c \quad (2)$$

$$\therefore dr = (dT - dt) c \quad (2')$$

From (1) & (2)', (1)

$$\therefore (dT - dt) c = \frac{rv}{r} dt$$

$$\therefore \frac{dT}{dt} = 1 + \frac{rv}{rc}$$

∴ Let P sends a train of waves towards O ,
and Consider two successive wave crests

Let $dT \equiv$ time interval between the two wave crests when received at O .

$dt \equiv$ time interval between the same wave crests when emitted by P . Classically

All this be for the outcome of Relativity.

(62)

Let λ_0 = wave length of emitted light

λ = " " " " received "

$$\frac{\lambda}{\lambda_0} = \frac{c dt}{c dT} = 1 + \frac{v}{rc} \quad (4)$$

a Classical Solution. } This is a

+ Note that the two intervals dT, dt are measured by obs. O.

→ Then, Consider another observer B at rest relative to P and moving with it.

+ Let dt' be the time interval between the two wave crests^{when} emitted by P, as measured by obs. B.

But according to S.R

$$dt = \frac{dt'}{\sqrt{1 - (v/c)^2}} \quad \text{[relativistically]} \quad (5)$$

$$dt = dt' / \sqrt{1 - (v/c)^2}$$

Dingle (63)

$$\frac{1}{dt} = \frac{\sqrt{1 - (v/c)^2}}{dt'} \Rightarrow \frac{dT}{dt} = \frac{dT}{dt'} \sqrt{1 - (v/c)^2} = (1 + \frac{xv}{rc})$$

$$\therefore \frac{dT}{dt'} = \frac{1 + \frac{xv}{rc}}{\sqrt{1 - (v/c)^2}}$$

Special
Relativity
Debates

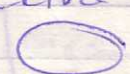
Herbert
Dingle
(6)

1922

+ Wiki
Internet

Google

Then if λ_0 = wavelength of emitted light as measured by B.

λ = wavelength of received light by 

$$\therefore \frac{\lambda}{\lambda_0} = \frac{c dT}{c dt'} = \frac{1 + \frac{xv}{rc}}{\sqrt{1 - (v/c)^2}}$$

But the radial Component of the velocity

$$\text{of Source } V_r = V \cos \theta = V \left(\frac{x}{r} \right) = \frac{xv}{r}$$

$$\therefore \frac{\lambda}{\lambda_0} = \frac{1 + V_r/c}{\sqrt{1 - (V/c)^2}}$$

(64)

$$\gamma(v) = 1 / \sqrt{1 - v^2/c^2}$$

Contraction

Paradoxes of Sp. Relat.

(1) Ehrenfest Paradox

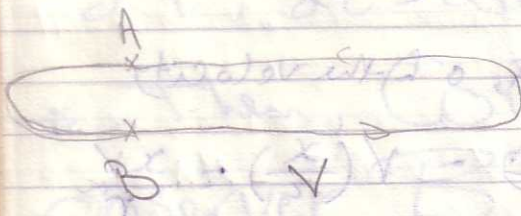
$$C = 2\pi r$$

Contracted

no contraction

Contraction

(2) Clock Paradox (by Einstein)



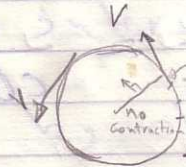
1st

$$T_A = T_B / \sqrt{1 - (v/c)^2}$$

$$T_A > T_B$$



No Contraction



Contraction

acceleration

2nd

$$T_B > T_A \quad (2)$$

S.R. = The restricted theory of relativity



$V \equiv$ uniform const. velocity

A

B

General relativity

acceleration

Experimental Evidence of Special Relativity.

Historically

1. Variation of Mass with velocity - (1902-1903) $m = m_0 / \sqrt{1 - (v/c)^2}$

• Experimental work connected with radio active elements has shown that β -particles emitted by different radio active elements have different masses and different velocities.

In 1903 Abraham suggested that mass may change with velocity. He, even, suggested an empirical formula

$$m = \left(\frac{3}{4} m_0 / \alpha^2 \right) \cdot \left[\frac{1 + \alpha^2}{2 \alpha \log \left(\frac{1 + \alpha}{1 - \alpha} \right)} - 1 \right]$$

where $\alpha = v/c$.

Later when Einstein found his relation $m = m_0 / \sqrt{1 - (v/c)^2}$.

Buchler in 1909 verified that Einstein's relation fits exactly with all the known experimental data.

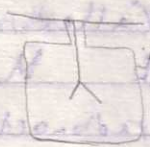
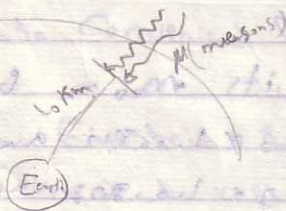


2. Time-Dilatation:

Cosmic rays Doctor engineer & photographer.

Adrian, Munich, Waly, Cosmic rays & (mesons)

μ -mesons (ALVAREZ) (1953) as stars & pyramids



It is well known that μ -mesons are formed at about 10^6 km/s from the Earth's surface, move with velocity $(0.995 c)$ for a certain time and finally decays into $\mu \rightarrow e + 2 \text{ neutrinos}$.

(66)

[The man riding with the particle]

In 1955 some German Physicists (Alvarez) tried to measure the life time of a μ -meson. They found it to be $t = 2.05 \times 10^{-6}$ sec. Therefore according to classical mechanics we have distance covered $= 2.05 \times 10^{-6} \times 0.99998 \times 10^8 \approx 600$ meter. But if we consider the proper life time of the μ -meson, then according to the time-dilatation of S.R. we have $t = t_0 / \sqrt{1 - (v/c)^2} \approx 10 \times t_0 = 2.05 \times 10^{-5}$ sec. hence this measured (previously) is t_0 of the man riding with the particle $\Rightarrow$ Distance Covered $= 10 \times 600 = 6000$ meter ≈ 6 Km. This result agrees with the fact that μ -mesons are observed at about 4 Km's from the Earth's surface.

3. Relation between Energy and mass ($E = mc^2$)

(i) Binding Energy?

If a nucleus consists of several particles then, it is said to be stable if its rest mass is less than the sum of the rest masses of its constituent particles, i.e. $M_0 < \sum m_0$.

$\Rightarrow \Delta M = \sum m_0 - M_0$ and its equivalent energy according to special relativity $\Delta E = c^2 \Delta M = c^2 (\sum m_0 - M_0)$

is called the binding energy

(ii) If $M_0 > \sum m_0$ the nucleus is said to be unstable or radioactive

examples Lithium ${}^6_3\text{Li}$ its mass ≈ 6.01697 amu

It consists of 3 protons $= 3 \times 1.00812$ amu

3 neutrons $= 3 \times 1.00893$ amu

$\Rightarrow \sum m_0 = 6.05115$ amu

$\Delta M = 6.05115 - 6.01697 = 0.03418$ amu $\approx 1.66 \times 10^{-27}$ gm. Now if

Energy equivalent to 1 amu ≈ 932 Mev.

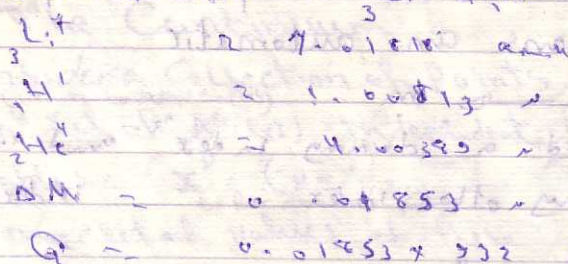
therefore Binding energy of ${}^6_3\text{Li}$ $\approx 0.03418 \times 932 \approx 31.8$ Mev

Marie-Antoinette Tonnelat

Former member of the International Committee on General Relativity and Gravitation

Special Relativity:

A Nuclear Reactions



General Relativity Lectures

presented By Professor Fahmy Mikhail

This result was verified experimentally in 1932 by Cockcroft & Walton

In the year 1990-1991

* TENSOR CALCULUS (B. Spain)

Tensor calculus.

(Levi Civita)

1. The Summation Convention.

We are going to consider sums of the form

$$S = a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n$$

where $a_1, x_1, x_2, \dots, x_n$ are n variable

Put in Convenient form (wanted)

To put this sum into Convenient form

$$S = \sum_{i=1}^n a_i x_i \quad (1)$$

by Einstein

Then Einstein suggested to drop the

Summation Sign $\sum$ and to replace the subscripts

in the Variables by superscripts and thus the sum (1)

will take the form $S = a_\alpha x^\alpha$ (2)

$$S = a_\alpha x^\alpha$$

*where the repeated index " α " [repeated once and twice only] (twice only)

(68)

Ex 1.2.1. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

where the repeated index " α " [repeated ~~index~~ and twice Only]
 one [repeated twice and twice ONLY, only UP ^{Variable}
 and one down] means an automatic

Summation from 1 to n

Such an index is called a "dummy index" and it
 can be changed to any other index

$$S = \sum_{\alpha=1}^n a_{\alpha} x^{\alpha} = \sum_{\beta=1}^n a_{\beta} x^{\beta} = \sum_{\gamma=1}^n a_{\gamma} x^{\gamma} \quad (\text{just a fact})$$

ex 1.1) Let $f = f(x^1, x^2, \dots, x^n)$

$$\Rightarrow df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \dots + \frac{\partial f}{\partial x^n} dx^n$$

$$\Rightarrow df = \frac{\partial f}{\partial x^{\alpha}} dx^{\alpha}$$

ex 2) The general quadratic expression in the nT

n variables $x^{\alpha} \quad (\alpha = 1, 2, \dots, n)$

$$ax^2 + bx + c = (ax + b)(cx + d) = \frac{ca}{a} x^2 + \frac{cb+da}{a} x + \frac{bd}{a}$$

$$S^2 = g_{11} (x^1)^2 + g_{12} x^1 x^2 + g_{13} x^1 x^3 + \dots + g_{22} (x^2)^2 + \dots + g_{33} (x^3)^2 + \dots + g_{nn} (x^n)^2$$

$$S^2 = \sum_{\alpha\beta} g_{\alpha\beta} x^{\alpha} x^{\beta}$$

total number of terms is

$$n + (n-1) = 2n-1$$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n$$

$$a'_1 x_1 + a'_2 x_2 + a'_3 x_3 + \dots + a'_n x_n$$

$$(a_1 a'_1) x_1^2 + (a_2 a'_2) x_2^2 + (a_3 a'_3) x_3^2 + \dots + (a_n a'_n) x_n^2$$

$$+ (a'_1 a_2) x_1 x_2 + (a'_2 a_1) x_2 x_1 + (a'_3 a_1) x_3 x_1 + \dots + (a'_n a_1) x_n x_1$$

$$+ (a'_1 a_3) x_1 x_3 + (a'_3 a_1) x_3 x_1 + (a'_2 a_3) x_2 x_3 + \dots + (a'_n a_3) x_n x_3$$

$$(a'_1 a_n) x_1 x_n + (a'_2 a_n) x_2 x_n + (a'_3 a_n) x_3 x_n + \dots$$

$$+ (a'_n a_n) x_n x_n$$

Tensors are the general if we were to reduce the kinds of vectors in $\mathbb{R}^3 \rightarrow$ it will work. (70)

S.R. (The whole laws of nature will be expressed by tensors)

1. Summation Convention - Dummy Indices.

2. The Continuum.

Consider a Collection of points, and each point is labeled by a set of n independent variables x^i ($i = 1, 2, 3, \dots, n$) (i.e. n-ordered tuples) Coordinate Like representation

S.t only set of values of these variables determine uniquely one and only one single point.

The number and nature of these variables are "arbitrary" (which is the purpose of inventing tensors)

+ It is always possible to change our way of labelling the points of the Continuum to any other way of labelling, $\bar{x}^i$ say, provided that (For observer B)

B wrt to A $\bar{x}^i = \bar{x}^i(x^1, x^2, x^3, \dots, x^n) \rightarrow (1)$

these functions are assumed to be Continuous, regular, differentiable, etc, most importantly, (Very good functions)

Solvable to get the inverse relations (i.e. A wrt to B)

A wrt to B $x^i = x^i(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) \rightarrow (2)$

* Such a change in our way of labelling the points of the Continuum is called a transformation of Coordinates and denoted by $x^i \rightarrow \bar{x}^i$

* Now now we take the lower case of tensors which is Vectors.

3. The Contravariant Vectors:

Consider a 4-dimensional Continuum whose points are determined by 4-variables x^i ($i = 1, 2, 3, 4$)

Then Consider the transformation $x^i \rightarrow \bar{x}^i$

where $\bar{x}^i = \bar{x}^i(x^1, x^2, x^3, x^4) \rightarrow (1)$

(i.e. B wrt to A)

Something Like a good Surface in 4-dimensional space $\Rightarrow$ Surface like is each $\bar{x}^i$

(71) $\frac{\partial \bar{x}^p}{\partial x^\alpha}$ a gradient like representation is if $\alpha = 1, 2, 3, 4 \rightarrow$ gradient is $\left[\frac{\partial \bar{x}^p}{\partial x^1}, \frac{\partial \bar{x}^p}{\partial x^2}, \frac{\partial \bar{x}^p}{\partial x^3}, \frac{\partial \bar{x}^p}{\partial x^4} \right]$

$$d\bar{x}^p = \sum_{\alpha=1}^4 \frac{\partial \bar{x}^p}{\partial x^\alpha} dx^\alpha \quad (2)$$

$$\boxed{d\bar{x}^p} = \sum_{\alpha=1}^4 \frac{\partial \bar{x}^p}{\partial x^\alpha} dx^\alpha = \frac{\partial \bar{x}^p}{\partial x^\alpha} dx^\alpha$$

[1] Def. 1 Any set of 4 quantities A^1, A^2, A^3, A^4 which transform under the transformation $x^\alpha \rightarrow \bar{x}^p$ according to the law:

[Line (2)]

$$\bar{A}^p = \sum_{\alpha=1}^4 \frac{\partial \bar{x}^p}{\partial x^\alpha} A^\alpha = (\text{gradient}) \cdot (\text{vector}) \quad A = [A^1, A^2, A^3, A^4] \quad \bar{A} = [\bar{A}^1, \bar{A}^2, \bar{A}^3, \bar{A}^4]$$

Constitute a "Contravariant Vector"

Note that relations (I) (4 in number) express each of the new elements

$\bar{A}^p$ ($p = 1, 2, 3, 4$) Linearly

in terms of the Old 4 ones:

e.g. taking $p = 3$ for example

$$\bar{A}^3 = \frac{\partial \bar{x}^3}{\partial x^\alpha} A^\alpha = \sum_{\alpha=1}^4 \frac{\partial \bar{x}^3}{\partial x^\alpha} A^\alpha \quad (V-imp)$$

Remember (True laws of nature)

Hence if the Vector (tensor) vanishes in any Coordinate System, it will Continue to vanish in any other Coordinate System.

i.e. if A and B are tensors st $A = B \rightarrow$

$$A - B = 0 \text{ is also a tensor}$$

4. Substitution Operator - Kronecker Deltas.

① Consider the expression

$$\frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} = \sum_{\alpha=1}^4 \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} =$$

$$\frac{\partial \bar{x}^{\nu}}{\partial \bar{x}^{\mu}} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}$$

But $\bar{x}^{\nu}$, $\bar{x}^{\mu}$ belong to the same System and there are totally

Independent,

Therefore.

$$\left\{ \begin{array}{l} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} = 0 \quad \text{if } \mu \neq \nu \\ = 1 \quad \text{if } \mu = \nu \end{array} \right.$$

This expression is denoted by

$$\frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} = \sum_{\mu} \delta_{\mu}^{\nu} \quad \left\{ \begin{array}{l} = 0 \quad \text{if } \mu \neq \nu \\ = 1 \quad \text{if } \mu = \nu \end{array} \right.$$

(ie ν w.r to μ)

$$\therefore \delta_{\alpha=\mu} = \delta_1 + \delta_2 + \delta_3 + \delta_4 = 4$$

this is a Summation because of the dummy variable α

Note also.

This happens only when

ν becomes equal to μ ie $\nu = \mu = \alpha$

$$\sum_{\nu=1}^4 \left[\sum_{\mu=1}^4 \left[\sum_{\alpha=1}^4 \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \right] \right] = \sum_{\alpha=1}^4 \delta_{\alpha}$$

(73)

Note also that

$$\sum_{\alpha=1}^4 \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \cdot \frac{\partial \bar{x}^\alpha}{\partial x^\alpha} = \frac{\partial x^\beta}{\partial x^\alpha} = \delta_\alpha^\beta$$

$\begin{cases} = 0 & \text{if } \alpha \neq \beta \\ = 1 & \text{if } \alpha = \beta \end{cases}$
 (ie β w.r.t. α)

(ii) Also if δ_α^β acts on any expression having the index α it changes it automatically to β and vice versa. Since δ_α^β is not a one term

$$\delta_\alpha^\beta A^\alpha = \sum_{\alpha=1}^4 \delta_\alpha^\beta A^\alpha = A^\beta$$

Contravariant vector

$$\Rightarrow \delta_\alpha^\beta A^\alpha = \sum_{\alpha=1}^4 \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \bar{A}^\beta$$

and vice versa

$$\delta_\alpha^\beta B_\beta = B_\alpha$$

Covariant vector

$$\delta_\mu^\nu T^{\mu\lambda} = T^{\nu\lambda}$$

$$\delta_\beta^\alpha T_{\lambda\alpha} = T_{\lambda\beta}$$

For this reason the Kronecker deltas are called

Substitution Operators

$$\delta_\alpha^\beta B_\beta = \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \cdot \frac{\partial \bar{x}^\alpha}{\partial x^\beta} B_\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\alpha} B_\alpha = \bar{B}_\alpha = B_\alpha$$

5. Invariants - (Scalars)

((Chaos dominate $\Leftrightarrow$ Invariants loose their Invariant Properties...))

In Certain Circumstances, we require to

attach to the points in certain region in a Continuum a certain property which doesn't change under any transformation of Coordinates $x^\nu \rightarrow \bar{x}^\nu$

Such a property is described by an "Invariant Function"

$\Phi(x^\nu)$ Say, such that

$$\bar{\Phi}(\bar{x}^\nu) = \Phi(x^\nu)$$

i.e. $\bar{\Phi}(\bar{x}) = \Phi(x)$

* The gradient of ~~this~~ an Invariant function is given by

one $\rightarrow \frac{\partial \Phi}{\partial x^\nu} \quad (\nu = 1, 2, 3, 4)$

Under the transformation $x^\nu \rightarrow \bar{x}^\nu$, we have

one $\rightarrow \frac{\partial \bar{\Phi}(\bar{x})}{\partial \bar{x}^\nu} = \frac{\partial \Phi}{\partial x^\alpha} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\nu}$

(God)
G has no gradient even though G is invariant

?? is this a correct statement ??

$$\frac{\partial \phi}{\partial x^\mu} = \left[\frac{\partial x^1}{\partial \bar{x}^\mu}, \frac{\partial x^2}{\partial \bar{x}^\mu}, \frac{\partial x^3}{\partial \bar{x}^\mu}, \frac{\partial x^4}{\partial \bar{x}^\mu} \right] \cdot \left[\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \frac{\partial \phi}{\partial x^3}, \frac{\partial \phi}{\partial x^4} \right]$$

(75) here it may not be 1 |L|=1 $\Rightarrow$ ds is similar to $\partial \bar{x}^\mu$ gradient of $\phi(x^1, x^2, x^3, x^4)$

$$\frac{\partial \phi}{\partial \bar{x}^\mu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial \phi}{\partial x^\alpha}$$

one something like directional derivative four

the μ th component only

6. Covariant Vectors:

Def: Any Set of 4 quantities

B_1, B_2, B_3, B_4

which transform under the

transformation $x^\mu \rightarrow \bar{x}^\mu$

by the law: (similar to (1) above)

$$\bar{B}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} B_\alpha \quad \text{--- (II)}$$

one four

$\mu = 1, 2, 3, 4$

Constitute a Covariant Vector.

Also $B_\mu = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \bar{B}_\alpha$

Note that relation II express each of the new elements

$$\bar{B}_\nu \quad (\nu = 1, 2, 3, 4)$$

Linearly in terms of the old 4-ones

e.g. taking $1=2 \Rightarrow$

$$\bar{B}_2 = \frac{\partial x^\alpha}{\partial \bar{x}^2} B_\alpha$$

α dummy index with a one term but 4 terms

$$= \sum_{\alpha=1}^4 \left[\frac{\partial x^\alpha}{\partial \bar{x}^2} \right] B_\alpha$$

something like basis

V.V. Imp

7. The Inner Product of any Contravariant Vector and a Covariant Vector is Invariant.

Let A^μ, B_ν be any

Cont. Var. and Cov. Vectors, respectively.

Hence under the transformation

$$x^\mu \rightarrow \bar{x}^\mu, \text{ we have}$$

(77)

$$\bar{A}^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} A^\alpha \quad \text{--- (1) Contr. vector}$$

$$\bar{B}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} B_\alpha \quad \text{--- (2) Cov. vectors}$$

Hence we want to prove that

$$\bar{A}^\mu \bar{B}_\mu = A^\mu B_\mu$$

$$\text{L.H.S} = \left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha} A^\alpha \right) \left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu} B_\alpha \right) =$$

Here is a Crime is committed

because the dummy variable (α) repeated more than twice

since dummy $\Rightarrow$

$$\text{L.H.S} = \bar{A}^\mu \bar{B}_\mu = \left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha} A^\alpha \right) \left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu} B_\alpha \right)$$

$$= \left(\frac{\partial \bar{x}^\mu}{\partial x^\alpha} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \right) \cdot A^\alpha B_\alpha =$$

$$(S^\alpha_\alpha A^\alpha) B_\alpha = A^\alpha B_\alpha = A^\mu B_\mu$$

show that $\bar{x}$ (dummy Variable)

8. Tensors

(i) A Contravariant tensor of the 2nd order

$$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \cdot \frac{\partial \bar{x}^\nu}{\partial x^\beta} T^{\alpha\beta}$$

$\Rightarrow \frac{\partial \bar{x}^\mu}{\partial x^\alpha} T^{\alpha\nu} = \bar{T}^{\mu\nu}$

(ii) A Covariant tensor of the 2nd order is defined by :

$$\bar{T}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \cdot \frac{\partial x^\beta}{\partial \bar{x}^\nu} C_{\alpha\beta}$$

$\frac{\partial x^\alpha}{\partial \bar{x}^\mu} C_{\alpha\nu} = \bar{T}_{\mu\nu}$

(iii) A mixed tensor of the 2nd order is defined by :

$$\bar{T}^\nu_\mu = \frac{\partial \bar{x}^\nu}{\partial x^\lambda} \cdot \frac{\partial x^\gamma}{\partial \bar{x}^\mu} T^\lambda_\gamma$$

$$\bar{B}^\epsilon_{\mu\nu\sigma} = \frac{\partial x^\lambda}{\partial \bar{x}^\mu} \cdot \frac{\partial x^m}{\partial \bar{x}^\nu} \cdot \frac{\partial x^n}{\partial \bar{x}^\sigma} \cdot \frac{\partial \bar{x}^\epsilon}{\partial x^k} B^k_{lmn}$$

4 × 4 × 4 × 4 (Number of terms)

Component

(79)

Note that Vector are tensors

of the 1st order.

& In Variants are of the

Zero Order $\bar{\Phi} \equiv \Phi$

here $\bar{\Phi}$ might have imaginary space i.e. of $(i) = 15$

$$\bar{B}_{\mu\nu\sigma}^E = \frac{\partial x^l}{\partial \bar{x}^\mu} \cdot \frac{\partial x^m}{\partial \bar{x}^\nu} \cdot \frac{\partial x^n}{\partial \bar{x}^\sigma} \cdot \frac{\partial \bar{x}^E}{\partial x^k} B_{lmn}^k$$

256 = $4 \times 4 \times 4 \times 4$ independent to

the position of E can be filled in 4 different ways

i.e. $E = 1, 2, 3$ or 4 , also the same for each of μ, ν, σ

Hence corresponding to the 4-methods of filling the position of E , there is also 4-methods of filling the position of μ hence there is

$4 \times 4 = 16$ method of filling the position of both E and μ

But corresponding to each method of the 16, there is 4-methods of filling the position of $\nu \Rightarrow$ total No. of methods of filling the position

of E, μ, ν is $4 \times 4 \times 4 = 16 \times 4 = 64$ method.

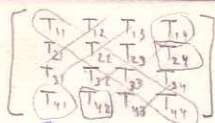
Also corresponding to each method of the 64, there is 4-methods of filling the position of $\sigma \Rightarrow$ total No. of "independent methods" of

filling the positions of E, μ, ν and σ is $4 \times 4 \times 4 \times 4 = 64 \times 4 = 256$ method.

$$\frac{\partial \bar{x}^E}{\partial x^k} B_{lmn}^k = B_{lmn}^E ; \frac{\partial x^n}{\partial \bar{x}^\sigma} B_{lmn}^E = B_{lmo}^E ;$$

$$\frac{\partial x^m}{\partial \bar{x}^\nu} B_{lmo}^E = B_{l\nu o}^E ;$$

$$\frac{\partial x^l}{\partial \bar{x}^\mu} B_{l\nu o}^E = \bar{B}_{\mu\nu o}^E$$



4 of the diagonal (main) + 6 of the other diagonals = 10 independent elements of a Symmetric Tensor

objects are 1, 2, 3, 4, i (The principal) where i is arbitrary, hence we have 10 n=5 objects that wanted to be arranged in two's without regard to position. $\frac{5!}{2!} = 120$ i may be 1, 2, 3, 4.

9. Symmetric and Skew Symmetric Tensors (80)

The 2nd order Cov. tensors $T_{\mu\nu}$ is said to be Symmetric w.r to the two indices μ, ν if

$$\text{Symmetric} \Rightarrow T = T^T$$

$$T_{\mu\nu} = T_{\nu\mu} \quad ; \quad T_{12} = T_{21} \quad ; \quad T_{34} = T_{43}$$

Hence Symmetric tensor has a 10 independent components in 4. dim space $\frac{n(n+1)}{2}$ the wings how; not why, is make it?

The How, $\frac{n+1}{2} = \frac{n+1}{2} \cdot 2$ $\frac{(n+1)(n)}{2 \times 2!}$; what is the principal?

* It is said to be Skew Symmetric if $T_{\mu\nu} = -T_{\nu\mu} \quad ; \quad T_{23} = -T_{32}$

Antisymmetric $\Rightarrow T = -T^T$

In this case it has (6) independent elements $= \frac{n(n-1)}{2}$ (in general)

$$n_c = \frac{n!}{(n-2)! 2!} = \frac{(n-2)!(n-1)n}{(n-2)! 2!}$$

$$\text{Since } T_{\mu\nu} = 0$$

* In General, any non symmetric $C_{\mu\nu}$ can be split into two parts, one Symmetric and the other Skew-Symmetric.

$$C_{\mu\nu} = \frac{1}{2} (C_{\mu\nu} + C_{\nu\mu}) + \frac{1}{2} (C_{\mu\nu} - C_{\nu\mu}) = \underline{C_{\mu\nu}} + \underline{C_{\mu\nu}}$$

where $\underline{C_{\mu\nu}} = \frac{1}{2} (C_{\mu\nu} + C_{\nu\mu})$ represents the Symmetric part.

and $\underline{C_{\mu\nu}} = \frac{1}{2} (C_{\mu\nu} - C_{\nu\mu})$ is the Skew part.

(81) (in areal sense) ?

10. Contraction of Tensors:

It is always possible to derive a tensor of a lower order (Lower by 2) i.e. if 4th order $\rightarrow$ reduced to 2nd

From another tensor of a higher order.

This process is known as the Contraction of tensors.

e.g. $B_{\mu\nu}^{\alpha\beta}$ of the 4th order, may give rise

to a tensor of the 2nd order by setting $\epsilon = \delta$ (say)

i.e. we can derive from it the 2nd order tensor defined by $C_{\mu\nu} = B_{\mu\nu}^{\alpha\alpha}$ a 2nd order tensor.

Proof

Under the transformation $x^\mu \rightarrow \bar{x}^\mu$

$$\bar{C}_{\mu\nu} = \bar{B}_{\mu\nu}^{\alpha\alpha} = \left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \cdot \frac{\partial x^\beta}{\partial \bar{x}^\nu} \right) \left(\frac{\partial x^\alpha}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^\beta}{\partial \bar{x}^\beta} \right) B_{\alpha\beta}^{\alpha\beta}$$

$$\begin{aligned} \bar{C}_{\mu\nu} &= \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \cdot \frac{\partial x^\beta}{\partial \bar{x}^\nu} \delta^{\alpha\beta} B_{\alpha\beta}^{\alpha\beta} \\ &= \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\nu} B_{\alpha\beta}^{\alpha\beta} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\alpha}{\partial \bar{x}^\nu} \cdot C_{\alpha\beta} \end{aligned}$$

$$\therefore C_{\mu\nu} = B_{\mu\nu}^{\alpha\alpha} = \text{Cov. tensor of 2nd order.}$$

* (Contravariant properties cancel, Cov. properties) *

Rule 1 If an upper index and a lower index are the same, the corresponding Contr. & Cov. Properties will cancel each other.

$$T_{\mu}^{\mu} = \text{Invariant} = \sum_{\mu=1}^4 T_{\mu}^{\mu} = T$$

The Scalar tensor or the trace of the tensor.

in general $A_\mu T^{\mu\nu} \equiv C^\nu$ Cont. var tensor

also, $A_\mu T^{\mu\beta} = C^\beta$ 2nd order tensor.

$\partial_\mu T^{\mu\nu} = I$ (invariant) (0 order tensor)

(how to know if it is a tensor or not)

($\exists +, -, \dots$ of tensors but $\neq \nabla$)

II. The Quotient Theorem!

If the inner product of an arbitrary vector with an unknown expression is a Tensor, then this expression will ~~be~~ itself be a tensor.

ie if $B(\mu\nu)$ [is an unknown expression] and A_μ is an arbitrary Covariant vector, such that,

$$A_\mu B(\mu\nu) = C^\nu \text{ (a Contravariant vector)}$$

then $B(\mu\nu)$ will be a tensor

$\Rightarrow$ we can find its order, kind (Cov. & Contr. var), symmetric, etc.

Proof (Standard steps)

Under the transformation $x^\mu \rightarrow \bar{x}^\mu$, we have

$$\bar{A}_\mu \bar{B}(\mu\nu) = \bar{C}^\nu$$

Vector at least, Contr. var. vector

(83)

$\Rightarrow$ Since $\bar{C}^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\alpha} C^\alpha$ but from (1)

dummy variable is used $\Rightarrow$ dummy

$$C^\alpha = A_\mu B(\mu\alpha)$$

$$\bar{A}_\mu \bar{B}(\mu\nu) = \frac{\partial \bar{x}^\nu}{\partial x^\alpha} A_\mu B(\mu\alpha) \quad (2)$$

We want A 's removed and remains the B 's

But

$$A_\mu = \frac{\partial \bar{x}^\mu}{\partial x^\lambda} \bar{A}_\mu \quad (3)$$

From substituting into (2) we get

$$\bar{A}_\mu \bar{B}(\mu\nu) = \frac{\partial \bar{x}^\nu}{\partial x^\alpha} \cdot \frac{\partial \bar{x}^\mu}{\partial x^\lambda} \bar{A}_\mu B(\mu\alpha) \quad (4)$$

no division, because index is not one but μ is relative

$$\bar{A}_\mu \left[\bar{B}(\mu\nu) - \frac{\partial \bar{x}^\nu}{\partial x^\alpha} \cdot \frac{\partial \bar{x}^\mu}{\partial x^\lambda} B(\mu\alpha) \right] = 0 \quad (5)$$

But since A_μ is arbitrary

$$\therefore \bar{B}(\mu\nu) = \frac{\partial \bar{x}^\nu}{\partial x^\alpha} \cdot \frac{\partial \bar{x}^\mu}{\partial x^\lambda} B(\mu\alpha)$$

$$\therefore B(\mu\nu) = B^{\mu\nu} \equiv$$

Cont. var tensor of the 2nd order.

This $\bar{A}_\mu$ is the good for us.

The Calculus part of tensors.

$$\begin{array}{l} ax + by + cz = 0 \\ \text{(arbitrary)} \\ x=0 \end{array}$$

$$\begin{array}{l} a_1x + a_2y + a_3z = 0 \\ b_1x + b_2y + b_3z = 0 \\ c_1x + c_2y + c_3z = 0 \end{array}$$

Homogeneous system of linear equations has the trivial solution (0) ;
[or] infinity of solution when $|D| = 0$

$$\text{(iii) if } |D| \neq 0 \Rightarrow x=y=z=0$$

$$Ax = 0$$

(i) $|A| = 0$ infinity of solutions

(ii)

12. Coordinate Partial derivatives of a vector:

Consider a Covariant vector

$A_\mu(x^\nu)$, then under the transformation $x^\nu \rightarrow \bar{x}^\nu$ we have

$$\bar{A}_\mu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} A_\alpha$$

Differentiating w.r to $\bar{x}^\nu$ we get

$$\frac{\partial \bar{A}_\mu}{\partial \bar{x}^\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \left(\frac{\partial A_\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \right) + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\mu \partial \bar{x}^\nu} A_\alpha =$$

$$\underbrace{\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \frac{\partial A_\alpha}{\partial x^\beta}}_{\text{Good Part}} + \underbrace{\frac{\partial^2 x^\alpha}{\partial \bar{x}^\mu \partial \bar{x}^\nu} A_\alpha}_{\text{bad part}}$$

This shows that the 16 quantities $\frac{\partial \bar{A}_\mu}{\partial \bar{x}^\nu}$ ~~do not~~

Do NOT form a tensor,

owing to the last term on the R.H.S

i.e. Coordinate partial differentiation do not give rise

to tensors. For this reason we must look for another type of differentiation which will give

rise to tensors.

(85)

Question 1, Although although $\frac{\partial A_k}{\partial x^j}$ isn't a tensor

Prove that $\frac{\partial A_{\mu\nu}}{\partial x^\nu} - \frac{\partial A_{\nu\mu}}{\partial x^\mu}$ is a Cov. tensor of the 2nd order. (All this is v. imp for physics) #

Riemannian SPaces.

(1) Cortes (Continuum) Labels

(iii) the metric of the space

1. The Metric of the Space:

In the Classical Euclidean Space we have the interval between any two neighbouring points is given by

$$\sigma_S^2 = \sigma_x^2 + \sigma_y^2$$

Coordinates (x, y)

$$\vec{s}^2 = \vec{x}^2 + \vec{y}^2 + \vec{z}^2$$

 (x, y, z)

In polar coordinates also Pythagorean

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (r, \theta, \phi)$$

$$ds^2 = dr^2 + r^2 d\phi^2$$

(more generally)

$$ds^2 = E du^2 + 2H du dv + F dv^2$$

2nd order (increments)

E, H, F are functions of (u, v)

Then Riemann generalized Euclidean geometry in two main respects.

(11)

- (1) Each point in the Continuum is ~~defined~~ Labeled by a set of independent n variables

$$x^\mu \quad (\mu = 1, 2, \dots, n)$$

(not as such)

(not as such)

- (2) The (interval) between two neighbouring points

$(x^\mu), (x^\mu + dx^\mu)$ is defined by the general quadratic expression in the increments dx^μ

i.e.
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (I)$$

$$\mu, \nu = 1, 2, 3, \dots, n$$

(symmetric must)

((if not symmetric the non-symmetric part will vanish))

$$ds^2 = g_{11} (dx^1)^2 + 2g_{12} (dx^1)(dx^2) + \dots$$

$$+ g_{22} (dx^2)^2 + \dots + g_{nn} (dx^n)^2$$

(I) is called the metric of the Riemannian Space.

The Coefficients $g_{\mu\nu}$ form a Covariant tensor

of the 2nd order, it is called the metric tensor or the fundamental tensor

(8.7)

It is a Symmetric tensor i.e. $g_{\mu\nu} = g_{\nu\mu}$

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{i.e.} \quad g_{23} = g_{32}$$

otherwise is nonsense, why! (a tensor)

N.B. If $A_{\mu\nu}$ is a Symmetric tensor &

$F^{\mu\nu}$ is a Skew-Symmetric tensor, then

$$A_{\mu\nu} F^{\mu\nu} = 0$$

because
 $F^{\mu\nu} = -F^{\nu\mu}$

V.I.M.P.

~~$A_{12} F^{12}$~~

~~$+ A_{21} F^{21}$~~

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

invariant

Contravariant

2. The Conjugate Metric Tensor

Consider a 4-dimensional Riemannian Space (R_4)

whose metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (\mu, \nu = 1, 2, 3, 4)$$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix} \neq 0$$

Then we introduce the new quantities

$g^{\mu\nu}$ defined by

$$g^{\mu\nu} = \frac{\text{Co-factor of } g_{\mu\nu} \text{ in } g}{g}$$

- (1) Determinant
(2) minor
(3) Co-factor

For example: $[g^{23}] = [\text{Co-factor of } g_{23} \text{ in } g] / g =$

$$= \frac{\begin{vmatrix} g_{11} & g_{12} & g_{14} \\ g_{21} & g_{22} & g_{24} \\ g_{31} & g_{32} & g_{34} \end{vmatrix}}{g}$$

$$\text{Hence } g_{\mu\alpha} g^{\nu\alpha} = 1 \quad \text{if } \mu = \nu$$

$$= 0 \quad \text{if } \mu \neq \nu$$

$$* g_{\mu\alpha} g^{\nu\alpha} = g_{\mu}^{\nu} = \sum_{\alpha} g_{\mu\alpha} g^{\nu\alpha} = 0 \quad \text{if } \mu \neq \nu$$

$$= 1 \quad \text{if } \mu = \nu$$

(Summation over α)

The new quantities $g^{\mu\nu}$ form a 2nd order Contravariant tensor, called the Conjugate metric tensor.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

* perform two v.i.m.t processes called Lowering and Raising of indices.

Question: Knowing that ds is an invariant prove that $g_{\mu\nu}$ appearing in the metric $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ is a Covariant tensor of the 2nd order? use (11)

(89)

3. Lowering and Raising of Indices: Associate Tensors:

* The metric tensor $g_{\mu\nu}$ and its Conjugate $g^{\mu\nu}$ enable us to perform two important processes:

(1) The lowering of an index:

given A^μ we can reach it to appear as

$$(i) A^\mu g_{\mu\nu} = A_\nu \rightarrow \boxed{A \text{ itself}}$$

$$(ii) T^{\mu\nu} g_{\nu\alpha} = T^\mu_\alpha$$

$$(iii) T^{\mu\nu} g_{\alpha\mu} g_{\nu\beta} = T^\mu_{\alpha\beta}$$

(2) The Raising of an index:

$$(i) B_\mu g^{\mu\nu} = B^\nu$$

$$(ii) T_{\mu\nu} g^{\mu\beta} = T^\beta_\nu$$

$$(iii) T_{\mu\nu} g^{\mu\beta} g^{\nu\epsilon} = T^{\beta\epsilon}$$

Tensors obtained from one another by the process of lowering or raising of indices are called "associate tensors."

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} dx^\alpha \cdot \frac{\partial x^\nu}{\partial x^\beta} dx^\beta \quad (i), \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (ii)$$

but ds is an invariant $\Rightarrow ds^2 = ds^2 \Rightarrow$

$$(i) - (ii) = 0 \Rightarrow g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} dx^\alpha \cdot \frac{\partial x^\nu}{\partial x^\beta} dx^\beta - g_{\mu\nu} dx^\mu dx^\nu = 0$$

$$\text{or equivalently, } g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} dx^\alpha dx^\beta - g_{\mu\nu} dx^\mu dx^\nu = 0 \quad \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} dx^\alpha dx^\beta - dx^\mu dx^\nu = 0$$

$$dx^\mu dx^\nu \left(\frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} - g^{\mu\nu} \right) = 0$$

since $dx^\mu, dx^\nu \neq 0 \Rightarrow$

$$\frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} = g^{\mu\nu}$$

$$g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} = g_{\alpha\beta}$$

i.e. by using the method

of changing the

dummy variable.

$g_{\mu\nu}$ is a Cov. tensor that's why

(بعضی اوقات در بعضی موارد)

Principle of best
curve path
Least action principle
is least power energy possible.
(as $\gamma = \text{space}$, $L = \text{time}$)

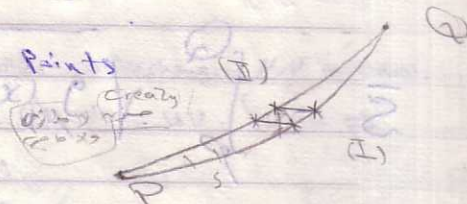
(90)

4. G codesics
Consider a 4-dimensional Riemannian space
whose metric is given by

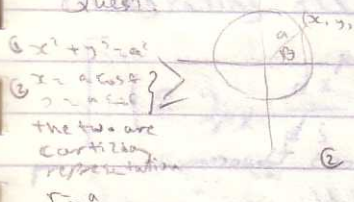
$$II \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \mu, \nu = 1, 2, 3, 4 \quad \text{same for (2)} \quad (2)$$

Consider two points P, Q in this space.

Let Curve (I) joins these two points
its eqⁿ is generally taken to be



Quest.



② is better for it is parametric (6)

For Curve (I) $x^\nu = x^\nu(s)$ $\nu = 1, 2, 3, 4$ (2) best representation
where "s" is the arcual length measured along the Curve.

Consider another Curve II very close to (I)
and passing through P, Q .
Its equation will be given by

$$\bar{x}^\nu = x^\nu + \epsilon \lambda^\nu \quad \text{memorized} \quad (3)$$

where ϵ is a very small quantity (Const.)

λ^ν is an arbitrary vector st

$$(\lambda^\nu)_P = 0, (\lambda^\nu)_Q = 0 \quad (4)$$

$$\text{Let } S = \int_P^Q ds, \quad \bar{S} = \int_P^Q d\bar{s}$$

be the length of the PQ arc measured along
Curve (I) and Curve (II) respectively.

(91)

2] Then Curve (I) is a geodesic if

$$\delta S = \bar{S} - S = 0$$

(all δx^μ in S)

$$\text{but } \bar{S} = \int_P^Q \bar{dS} = \int_P^Q \sqrt{g_{\mu\nu}(\bar{x}) \frac{d\bar{x}^\mu}{ds} \frac{d\bar{x}^\nu}{ds}} ds$$

taken along Curve (II)

(Same algebra)

Ess. 5.2.6

$$\bar{S} = \int_P^Q \sqrt{g_{\mu\nu}(\bar{x}) \frac{d\bar{x}^\mu}{ds} \cdot \frac{d\bar{x}^\nu}{ds}} ds$$

but from (3)

$$\frac{d\bar{x}^\mu}{ds} = \frac{dx^\mu}{ds} + \epsilon \frac{d\lambda^\mu}{ds}$$

["Taylor's theorem"]

$$\frac{d\bar{x}^\nu}{ds} = \frac{dx^\nu}{ds} + \epsilon \frac{d\lambda^\nu}{ds}$$

$$f(x) = \sum_{j=0}^m \frac{f^{(j)}(a)(x-a)^j}{j!} + R_m(x)$$

3] Also from Taylor's theorem we have:

$$g_{\mu\nu}(\bar{x}) = g_{\mu\nu}(x) + \epsilon \lambda^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + O(\epsilon^2) \Rightarrow$$

$$\bar{S} = \int_P^Q \left[\left(g_{\mu\nu} + \epsilon \lambda^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \left(\frac{dx^\mu}{ds} + \epsilon \frac{d\lambda^\mu}{ds} \right) \left(\frac{dx^\nu}{ds} + \epsilon \frac{d\lambda^\nu}{ds} \right) \right]^{1/2} ds$$

$$\approx \int_P^Q \left[g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \epsilon \left\{ g_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\lambda^\nu}{ds} + g_{\mu\nu} \frac{d\lambda^\mu}{ds} \frac{dx^\nu}{ds} + \lambda^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right\} \right]^{1/2} ds$$

$$= (1,3 + 2,3 + 1,4 + 2,4) = 1,3,5 + \frac{2,3,5}{\epsilon} + \frac{1,4,5}{\epsilon} + \frac{2,4,5}{\epsilon^2} + \frac{1,3,6}{\epsilon} + \frac{2,3,6}{\epsilon^2} + \frac{1,4,6}{\epsilon^2} + \frac{2,4,6}{\epsilon^3}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$(1 + \epsilon)^n \approx 1 + n\epsilon$$

Since $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} ds^2 = 1$

$\Rightarrow [1 + \epsilon \{ (1/3, 6) + (1/4, 5) + (2/3, 5) \}]^{1/2} \approx 1 + \frac{1}{2} \epsilon \{ (1/3, 6) + (1/4, 5) + (2/3, 5) \}$

1st 2nd 3rd

Since, with 1/3, 6 and 1/4, 5 are not damaged by 1/4, 5 becomes $g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1$

Notice $g_{\mu\nu} = g_{\nu\mu}$

$$\approx \int_p^Q [1 + \epsilon \{ \dots \}]^{1/2} ds \approx \int_p^Q [1 + \frac{1}{2} \epsilon \{ \dots \}] ds \quad (92)$$

For a geodesic $ds = \bar{s} - s = 0$

$$\therefore \delta s = \int_p^Q [1 + \frac{1}{2} \epsilon \{ \dots \}] ds - \int_p^Q 1 ds$$

$$\delta = \frac{1}{2} \epsilon \int_p^Q \left[2g_{\mu\nu} \frac{dx^\mu}{ds} \left(\frac{d\lambda^\nu}{ds} \right) + \lambda^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right] ds = 0$$

Since, $g_{\mu\nu} = g_{\nu\mu}$ (Symmetric tensor) Using dummy Variable Method $\Rightarrow$ Interchanging μ, ν in (1/4, 5)

Integration by Parts of the 1st term above $\Rightarrow \int u dv = uv - \int v du \Rightarrow$

Integrating the 1st term by parts:

$$\left[\left(2g_{\mu\nu} \frac{dx^\mu}{ds} \right) \lambda^\nu \right]_p^Q - \int_p^Q \left[\lambda^\nu \frac{d}{ds} \left(2g_{\mu\nu} \frac{dx^\mu}{ds} \right) + \lambda^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right] ds = 0$$

From (4) Hence $\left[\left(2g_{\mu\nu} \frac{dx^\mu}{ds} \right) \lambda^\nu \right]_p^Q = 0$

(dummy variable)

$$\therefore \int_p^Q \lambda^\alpha \left[\frac{d}{ds} \left(2g_{\mu\nu} \frac{dx^\mu}{ds} \right) - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right] ds = 0$$

α is better to get the (dummy variable) let ν become α in the 1st term.

Common factor λ^α

Since λ^α is arbitrary hence we can make

Curve II, III, etc

every time we get another yet the curve II, III, etc by integration $= 0$

But since λ^α is an arbitrary vector

$$\therefore \left[\dots \right] = 0$$

mathematically

(93)

$$2g_{\mu\nu} \frac{d^2 x^\mu}{ds^2} + 2 \frac{dx^\mu}{ds} \cdot \frac{dg_{\mu\nu}}{ds} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \cdot \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = 0$$

returning α to be ν in the 1st term

$$\therefore \frac{d}{ds} \left(2g_{\mu\nu} \frac{dx^\mu}{ds} \right) - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \cdot \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = 0$$

$$= dg_{\mu\nu}/ds \rightarrow \text{put here } \nu \text{ as } \alpha \text{ and derivative w.r to } \nu \text{ (better to take a common factor)}$$

$$\therefore g_{\mu\nu} \frac{d^2 x^\mu}{ds^2} + \frac{dx^\mu}{ds} \cdot \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} \cdot \frac{dx^\alpha}{ds} \right) - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{ds} \frac{dx^\alpha}{ds} = 0$$

returning ν as α in the 1st term

$$\therefore g_{\mu\alpha} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left[2 \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right] \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = 0$$

we want to get rid of $g_{\mu\alpha}$, $\oplus$ division
multiplying by $g^{\alpha\sigma}$ (α is better), μ (other side)

④ or $g^{\alpha\sigma}$

$$\therefore g^{\alpha\sigma} g_{\mu\alpha} = \delta_{\mu}^{\sigma} = \begin{cases} 1 & (\sigma=\mu) \\ 0 & (\sigma \neq \mu) \end{cases}$$

We want it much symmetric,

we try; is it possible?

Since $g_{\mu\alpha} = g_{\alpha\mu}$ (Symmetric tensor).

$$\therefore \frac{d^2 x^\sigma}{ds^2} + \frac{1}{2} \sum_{\mu, \nu} g^{\alpha\sigma} \left[\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\mu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right] \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

Sensitive

(the summation here will be w.r to μ, ν or α ; for some σ .)

Finally we get

$$\frac{d^2 x^\sigma}{ds^2} + \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = 0 \quad (I)$$

giving the differential equations for $\sigma = 1, 2, 3, 4$
a geodesic

where

$$\left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\mu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right)$$

are the Christoffel 3-index Symbols
of the 2nd ~~xxx~~ kind.

$$\frac{d^2 x^0}{ds^2} + \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{d^2 x^0}{ds^2} + \sum_{\mu=1}^4 \left[\sum_{\nu=1}^4 \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} \frac{dx^\nu}{ds} \right] \frac{dx^\mu}{ds} = 0 \quad 0, 1, 2, 3, 4.$$

$$\frac{d^2 x^0}{ds^2} + \frac{1}{2} \sum_{\mu=1}^4 \sum_{\nu=1}^4 \left[\sum_{\alpha=1}^4 g^{\alpha\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \right] \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

physically a very int situation.

(94)

Note: we can easily show that equations (I) will ~~reduce~~ give rise to the classical equations of a straight line in the case of a Euclidean 3-Dimensional Space.

* The metric of this Space is

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (1)$$

$$\therefore g_{11} = 1 \quad ; \quad g_{22} = 1 \quad , \quad g_{33} = 1 \quad ,$$

$$g_{\mu\nu} = 0 \quad \text{if } \mu \neq \nu$$

$$\text{hence } \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) = 0$$

Therefore All the Christoffel Symbols $\left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} = 0$

$\therefore$ Eqn (I) will reduce to

$$\frac{d^2 x^0}{ds^2} = 0$$

$$\therefore \text{G21} \Rightarrow \frac{d^2 x^1}{ds^2} = 0 \Rightarrow \frac{dx^1}{ds} = \text{const.} = 1$$

$$\Rightarrow \text{another integration } x = \lambda s + a/T \quad (2)$$

$$\frac{d^2 x^0}{ds^2} + \frac{1}{2} \sum_{\mu=1}^4 \left[\sum_{\nu=1}^4 \left[\sum_{\alpha=1}^4 g^{\alpha\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \right] \frac{dx^\mu}{ds} \right] \frac{dx^\nu}{ds} = 0$$

$$\frac{d^2 x^0}{ds^2} + \frac{1}{2} \sum_{\mu=1}^4 \left[\sum_{\nu=1}^4 \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \right] \frac{dx^\nu}{ds} = 0$$

(95)

$$\frac{dx}{ds} = \frac{x-a}{l} \Rightarrow \frac{dx}{ds} = \frac{x-a}{l} \Rightarrow \frac{dx}{ds} = \frac{x-a}{l} \Rightarrow \frac{dx}{ds} = \frac{x-a}{l}$$

$$\frac{dy}{ds} = \frac{y-b}{m} \Rightarrow \frac{dy}{ds} = \frac{y-b}{m} \Rightarrow \frac{dy}{ds} = \frac{y-b}{m} \Rightarrow \frac{dy}{ds} = \frac{y-b}{m}$$

Eliminating "S" between these equations we get:

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = S$$

which are the standard classical eqns of

a straight line with actual direction

cosines (l, m, n). Since from (1)

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$$

$$1 = l^2 + m^2 + n^2$$

{ which Newton straight line body moves in
Einstein body moves on a geodesic }

+ The Christoffel 3-index Symbol:

There are two kinds of these symbols:

(i) The Christoffel Symbols of the 1st kind.

$$[X_{\mu\nu}, \alpha] = \frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right)$$

1st due to (α)

$$[l, m, k] = \frac{1}{2} \left(\frac{\partial g_{lk}}{\partial x^m} + \frac{\partial g_{mk}}{\partial x^l} - \frac{\partial g_{lm}}{\partial x^k} \right)$$

(iii) The Christoffel Symbols of the 2nd Kind:

$$\left\{ \begin{matrix} \sigma \\ \mu \nu \end{matrix} \right\} \overset{\text{(Remember raising)}}{\underset{\text{(not a tensor)}}{=}} \sum_{\alpha=1}^4 g^{\alpha\sigma} [\mu \nu, \alpha] \quad \text{2nd due to } \alpha \text{ & } \sigma.$$

$$\frac{1}{2} \sum_{\alpha=1}^4 g^{\alpha\sigma} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right)$$

Summation for α ??
for some σ ; μ, ν .

prove: $[\mu \nu, \alpha] + [\alpha \nu, \mu] = \frac{\partial g_{\mu\alpha}}{\partial x^\nu}$

proach: $\frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right)$

$$+ \frac{1}{2} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\mu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\nu}}{\partial x^\mu} \right) = \frac{\partial g_{\mu\alpha}}{\partial x^\nu}$$

+ Properties of $\left\{ \begin{matrix} \sigma \\ \mu \nu \end{matrix} \right\}$

1. They are Symmetric in μ and ν , they are 4x10 independent element.

No. of σ 's = 2
 $\sigma = 1, 2, 3, 4$ due to the symmetry of $\mu \nu$

2. They do not form a tensor, and have a special law of transformation.

$$3. [\mu \nu, \alpha] + [\alpha \nu, \mu] = \frac{\partial g_{\mu\alpha}}{\partial x^\nu}$$

(4 added together)

$$4. \left\{ \begin{matrix} \sigma \\ \mu \nu \end{matrix} \right\} = \frac{1}{2} g^{\sigma\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) =$$

$$\frac{1}{2} g^{\sigma\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \right) = \frac{1}{2} g^{\sigma\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^\mu} \quad \dots (1)$$

ie Interchanging $\nu, \alpha \Rightarrow$ since $g^{\nu\alpha} = g^{\alpha\nu}$ (Symmetric).

hence the result. [we did so in the 3rd only, for we want to show the

because both ν, α has been repeated twice,
they are dummy and can be interchanged.

Similar terms up,
so they cancel
each other

(97)

determinant

5. Since $g = \|g_{\mu\nu}\| \Rightarrow$ If we accept that

$$\frac{\partial g}{\partial x^\mu} = \frac{\partial g_{\nu\alpha}}{\partial x^\mu} \times (\text{Cofactor of } g_{\nu\alpha})$$

where $g^{\nu\alpha} = \text{Cofactor of } g_{\nu\alpha} / g \Rightarrow$

$$\text{Therefore, } \frac{\partial g}{\partial x^\mu} = \sum_{\nu, \alpha} \frac{\partial g_{\nu\alpha}}{\partial x^\mu} \times (g^{\nu\alpha} g)$$

$$\therefore \frac{1}{g} \cdot \frac{\partial g}{\partial x^\mu} = g^{\nu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^\mu} \quad (2)$$

From (1), (2) we get; after multiplying (2) by (1/2);

$$\left\{ \begin{matrix} \nu \\ \mu\nu \end{matrix} \right\} = \left(\frac{1}{2} g^{-1} \frac{\partial g}{\partial x^\mu} \right) = \frac{\partial \log \sqrt{g}}{\partial x^\mu} = \frac{\partial \log \sqrt{-g}}{\partial x^\mu}$$

$$\left\{ \begin{matrix} \nu \\ \mu\nu \end{matrix} \right\} = \frac{\partial \log \sqrt{\pm g}}{\partial x^\mu} \quad (3)$$

+ Notice, equivalently we could have taken $(\ln \sqrt{-g})$; $\log \equiv \ln$; here.

$$(3) = \frac{1}{\sqrt{-g}} \cdot \frac{1}{2} (-g)^{-\frac{1}{2}} \cdot \frac{\partial (-g)}{\partial x^\mu} = \frac{1}{2} g^{-1} \frac{\partial g}{\partial x^\mu}$$

$$+ \sum_{\nu=1}^4 \left[\left\{ \begin{matrix} \nu \\ \mu\nu \end{matrix} \right\} \right] \frac{dx^\nu}{ds} = \sum_{\nu=1}^4 \left[\sum_{\alpha=1}^4 g^{\nu\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \right] \frac{dx^\nu}{ds}$$

When $\alpha=1 \rightarrow$

as $\nu=1 \rightarrow$

$$\begin{matrix} \nu & \mu \\ \downarrow & \downarrow \\ 1 & 1 \\ \hline & 1 \\ (1+1) \end{matrix}$$

$$\begin{matrix} \nu & \mu \\ \downarrow & \downarrow \\ 1 & 1 \\ \hline & 1 \\ (-1) \end{matrix}$$

just a name
works for all.

S. Rela. 5. Covariant Differentiation:

1. Def. The Covariant derivative of vector A_μ w.r to(exam) i.e derivative of a vector A_μ w.r to the vector x^μ x^μ is defined by $A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{1}{2} \sum_{\alpha=1}^4 g^{\alpha\beta} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) A_\alpha$, $\mu, \nu = 1, 2, 3, 4$.

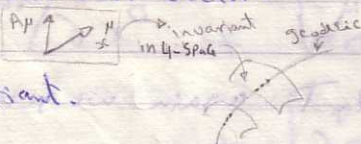
$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \left\{ \begin{matrix} \alpha \\ \mu \nu \end{matrix} \right\} A_\alpha \quad \text{--- (I)}$$

Prove that R.H.S of (I) is a tensor: i.e. " $A_{\mu;\nu}$ " is a tensor.Let A_μ be an arbitrary Cov. Vector.

(tangent)

But $\frac{dx^\mu}{ds}$ = Cont. var vector.

$$\therefore A_\mu \frac{dx^\mu}{ds} \equiv \text{invariant.}$$



remember

$$\frac{dR}{ds} = \nabla R \cdot \mathbf{T}, \quad \|\mathbf{T}\| = 1$$

$$(x(s), y(s), z(s)) = \mathbf{R}$$

 $\Rightarrow$ tangent to the curve is

$$\mathbf{T} = \frac{d\mathbf{R}}{ds}; \quad \frac{d\mathbf{T}}{ds} = \kappa \cdot \mathbf{N}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

+ then the rate of change of this invariant taken along any Curve is also invariant. (In fact the only Curve we have is the geodesic)

$$\text{i.e. } \frac{d}{ds} \left[A_\mu \frac{dx^\mu}{ds} \right] \equiv \text{invariant} = I$$

$$\therefore A_\mu \frac{d^2 x^\mu}{ds^2} + \frac{dx^\mu}{ds} \left(\frac{\partial A_\mu}{\partial x^\nu} \cdot \frac{dx^\nu}{ds} \right) = I$$

$$A_\sigma \frac{d^2 x^\sigma}{ds^2} + \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = I \quad \text{--- (1)}$$

Taking this rate of change along a geodesic:

$$\frac{d^2 x^\sigma}{ds^2} + \left\{ \begin{matrix} \sigma \\ \mu \nu \end{matrix} \right\} \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = 0 \quad \text{--- (2)}$$

i.e. we want to make use of (2) in (1).

(99)

$$\nabla A_0(x) \Rightarrow A_0 \frac{d^2 x^0}{ds^2} = -A_0 \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} \quad (3)$$

Substituting from (3) into (1) we get.

$$\frac{d}{ds} \left[A_0 \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} \right] + A_0 \frac{\partial A_\mu}{\partial x^\nu} \cdot \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = I$$

$$\left[\frac{\partial A_\mu}{\partial x^\nu} - \left\{ \begin{matrix} 0 \\ \mu\nu \end{matrix} \right\} A_0 \right] \cdot \frac{dx^\mu}{ds} \cdot \frac{dx^\nu}{ds} = I$$

$$= \left[A_{\mu;\nu} (dx^\mu/ds) \right] (dx^\nu/ds) = I$$

Therefore by Quotient theorem (Q.T) we get

$$A_{\mu;\nu} dx^\mu/ds = \text{Tensor, vector}$$

Hence using Q.T again we get

$$A_{\mu;\nu} = \partial A_\mu / \partial x^\nu - \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} A_\sigma \quad \text{dummy}$$

[γ is a Covariant tensor of the 2nd order.]

Rules of Covariant Diff

(1) The Covariant derivative of the Contra - Variant vector

B^μ w.r.to x^ν is given by:

$$B_{;\nu}^\mu = \frac{\partial B^\mu}{\partial x^\nu} + \frac{1}{2} \sum_{\alpha} \left[\sum_{\mu} g^{\mu\alpha} \left(\frac{\partial g_{\alpha\lambda}}{\partial x^\nu} + \frac{\partial g_{\mu\lambda}}{\partial x^\alpha} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right) \right] B^\alpha$$

(100)

$$B^{\mu}_{;\nu} = \partial B^{\mu} / \partial x^{\nu} + \{^{\mu}_{\alpha\nu}\} B^{\alpha} \quad (\text{Prove ?}).$$

(2) Covariant derivative of a tensor

* $T_{\mu\nu;\sigma} = \partial T_{\mu\nu} / \partial x^{\sigma} - \{^{\alpha}_{\mu\sigma}\} T_{\alpha\nu} - \{^{\alpha}_{\nu\sigma}\} T_{\mu\alpha}$

(ordinary derivative)

* $C_{lm;n} = \partial C_{lm} / \partial x^n - \{^k_{ln}\} C_{km} - \{^k_{mn}\} C_{lk}$

* $T^{\mu\nu}_{;\sigma} = \partial T^{\mu\nu} / \partial x^{\sigma} + \{^{\mu}_{\alpha\sigma}\} T^{\alpha\nu} + \{^{\nu}_{\lambda\sigma}\} T^{\mu\lambda}$

(3) * $T^{\nu\lambda}_{;\mu} = \partial T^{\nu\lambda} / \partial x^{\mu} - \{^{\epsilon}_{\mu\sigma}\} T^{\nu\lambda}_{;\epsilon} + \{^{\nu}_{\alpha\sigma}\} T^{\alpha\lambda}_{;\mu} + \{^{\lambda}_{\beta\sigma}\} T^{\nu\beta}_{;\mu}$

+ $\frac{1}{2} \frac{1}{\sqrt{|g|}} dS^2 = \frac{1}{2} \frac{1}{\sqrt{|g|}} g_{\mu\nu} dx^{\mu} dx^{\nu}$

$\{^{\sigma}_{\mu\nu}\} = g^{\sigma\alpha} [\mu\nu, \alpha] \Rightarrow$
 $g_{\sigma\alpha} \{^{\sigma}_{\mu\nu}\} = g_{\sigma\alpha} g^{\sigma\alpha} [\mu\nu, \alpha] = [\mu\nu, \alpha]$

(4) $g_{\mu\nu;\sigma} = \partial g_{\mu\nu} / \partial x^{\sigma} - \{^{\alpha}_{\mu\sigma}\} g_{\alpha\nu} - \{^{\beta}_{\nu\sigma}\} g_{\mu\beta}$

$\{^{\alpha}_{\mu\sigma}\} g_{\alpha\nu} = \frac{1}{2} g_{\mu\nu} g^{\alpha\lambda} \left[\frac{\partial g_{\mu\lambda}}{\partial x^{\sigma}} + \frac{\partial g_{\lambda\mu}}{\partial x^{\sigma}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right] =$

$\frac{1}{2} \delta^{\lambda}_{\nu} [\mu\sigma, \lambda] \cdot$ but $\delta^{\lambda}_{\nu} = \begin{cases} 1, & \lambda = \nu \\ 0, & \lambda \neq \nu \end{cases}$

$- \{^{\beta}_{\nu\sigma}\} g_{\mu\beta} = - \{^{\beta}_{\nu\sigma}\} g_{\mu\beta} = -I$

hence, as λ becomes ν the only bracket $[\mu\sigma, \lambda]$ that will survive is $[\mu\sigma, \nu]$.

$$\partial g_{\mu\nu} / \partial x^{\sigma} - [\mu\sigma, \nu] - [\nu\sigma, \mu] =$$

⊙ G.O. the Conjugate Metric Tensor

are we free to choose it as to go with $\partial g_{\mu\nu}$?

(101)

$$= \partial g_{\mu\nu} / \partial x^\alpha - \partial g_{\mu\nu} / \partial x^\alpha = 0$$

The Covariant derivatives of the metric tensor
(V.I.M.P.)
vanish identically.

ie $g_{\mu\nu}$ and its conjugate $g^{\mu\nu}$ behave as
Constants under Covariant differentiation.

(as a result of ⑧)

Hence we can lower & raise indices under Cov. diff.

(5) The Covariant derivative of an Invariant:

Let I be an invariant & A_μ be an
arbitrary Covariant vector.

$$(IA_\mu) = \text{Cov. vector.} \quad (E)$$

Taking the Covariant der. w.r. to x^ν , we get $(IA_\mu)_{;\nu}$

$$= \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \frac{1}{2} g^{\alpha\beta} \left[\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right]$$

$$(IA_\mu)_{;\nu} = \partial(IA_\mu) / \partial x^\nu - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} (IA_\alpha) =$$

$$I (\partial A_\mu / \partial x^\nu) + A_\mu (\partial I / \partial x^\nu) - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} (IA_\alpha) =$$

$$I (\partial A_\mu / \partial x^\nu) - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} A_\alpha + A_\mu \frac{\partial I}{\partial x^\nu}$$

but L.H.S

$$= I A_{\mu;\nu} + A_\mu \frac{\partial I}{\partial x^\nu} \quad (5)$$

but L.H.S = $I A_{\mu;\nu} + A_{\mu} I_{;\nu}$ (2)

from (1), (2) we get

$$I A_{;\nu} - \frac{\partial I}{\partial x^{\nu}} = I_{;\nu}$$

Solve $I = A_{\mu} B^{\mu}$; $I_{;\nu}$

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

$\left\{ \begin{matrix} \alpha \\ \mu \nu \end{matrix} \right\} \rightarrow \Gamma_{\mu\nu}^{\alpha}$

$\Gamma_{\mu\nu}^{\alpha} = 0$

Constant or not Constant
not Constant
if it is not constant
it is not constant

2. The Riemann - Christoffel Tensor:

(and its brother)

Curvature Tensor:

To find whether or not whether Covariant differentiation
whether Covariant differentiation
Commutes or does not Commute

Let A_{μ} be an arbitrary Cov. vector, then

$$A_{\mu;\nu} = \partial A_{\mu} / \partial x^{\nu} - \left\{ \begin{matrix} \alpha \\ \mu \nu \end{matrix} \right\} A_{\alpha}$$

(103)

remember, symmetric
in $\mu\sigma$ (*)

$$\{\alpha\}_{\mu\sigma} = \frac{1}{2} \delta^{\alpha\beta} \left(\frac{\partial g_{\mu\sigma}}{\partial x^\alpha} + \frac{\partial g_{\sigma\mu}}{\partial x^\beta} - \frac{\partial g_{\mu\beta}}{\partial x^\sigma} \right) = \frac{1}{2} \delta^{\alpha\beta} \frac{\partial g_{\mu\sigma}}{\partial x^\beta}$$

$$\{A_{\mu;\sigma}\}_{;\sigma} = \partial A_{\mu;\sigma} / \partial x^\sigma = \{\alpha\}_{\mu\sigma} A_{\alpha;\sigma}$$

we want as (*) $\{\alpha\}_{\mu\sigma} A_{\mu;\sigma} = - \{\alpha\}_{\mu\sigma} A_{\mu;\sigma} =$

$$\frac{\partial}{\partial x^\sigma} \left(\frac{\partial A_\mu}{\partial x^\sigma} - \{\alpha\}_{\mu\sigma} A_\alpha \right) - \{\alpha\}_{\mu\sigma} \left(\frac{\partial A_\alpha}{\partial x^\sigma} - \{\alpha\}_{\sigma\alpha} A_\sigma \right)$$
$$- \{\alpha\}_{\mu\sigma} A_{\mu;\sigma} =$$

old, symmetric. ω

$$A_{\mu;\sigma} = \left[\frac{\partial^2 A_\mu}{\partial x^\sigma \partial x^\sigma} - \{\alpha\}_{\mu\sigma} \frac{\partial A_\alpha}{\partial x^\sigma} - A_\alpha \{\alpha\}_{\mu\sigma} \right]$$
$$- \{\alpha\}_{\mu\sigma} \frac{\partial A_\alpha}{\partial x^\sigma} + \{\alpha\}_{\mu\sigma} \{\alpha\}_{\sigma\alpha} A_\sigma - \{\alpha\}_{\mu\sigma} A_{\mu;\sigma}$$

(1) (Change α by β in (3)) (2)

$$= \frac{\partial^2 A_\mu}{\partial x^\sigma \partial x^\sigma} - \{\alpha\}_{\mu\sigma} \frac{\partial A_\alpha}{\partial x^\sigma} - \{\alpha\}_{\mu\sigma} \frac{\partial A_\alpha}{\partial x^\sigma} - \{\alpha\}_{\mu\sigma} A_{\mu;\sigma}$$
$$- A_\sigma \left[\{\alpha\}_{\mu\sigma} - \{\alpha\}_{\mu\sigma} \{\alpha\}_{\sigma\alpha} \right] = A_{\mu;\sigma} \quad (I)$$

+ Similarly we get

interchange σ by ν

$$A_{\mu;\sigma\nu} = \frac{\partial A_\mu}{\partial x^\sigma \partial x^\nu} - \{\alpha\}_{\mu\sigma} \frac{\partial A_\alpha}{\partial x^\nu} - \{\alpha\}_{\mu\nu} \frac{\partial A_\alpha}{\partial x^\sigma}$$
$$- \{\alpha\}_{\sigma\nu} A_{\mu;\alpha} - A_\sigma \left[\{\alpha\}_{\mu\sigma} - \{\alpha\}_{\mu\sigma} \{\alpha\}_{\sigma\nu} \right] \quad (II)$$

$$(I) - (II) = A_{\mu;\nu\sigma} - A_{\mu;\sigma\nu} = A_\sigma \left[\{\alpha\}_{\mu\sigma} - \{\alpha\}_{\mu\sigma} \{\alpha\}_{\sigma\nu} \right]$$
$$+ \left[\{\alpha\}_{\mu\sigma} \{\alpha\}_{\sigma\nu} - \{\alpha\}_{\mu\nu} \{\alpha\}_{\sigma\sigma} \right] =$$

tensor
3rd order

A_σ (arbitrary vector)

$B_{\mu\nu\sigma}$

Consider a symmetric tensor w.r to a number of its indices; not all, and then consider how it can be symmetric w.r to all of its indices?

(104)

Hence, by the Q.T. we get, since L.H.S

= tensor A_β is an arbitrary vector.

Contracting β with α we get

$$B_{\mu\nu\sigma}^\beta = \left\{ \begin{matrix} \beta \\ \mu\sigma \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \beta \\ \mu\nu \end{matrix} \right\}_{,\sigma} + \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\nu \end{matrix} \right\}$$

$$- \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\sigma \end{matrix} \right\} \quad (\text{the two opposite sides are } +\text{ and } -)$$

is a fundamental tensor of the 4th order,

it is called the Riemann-Christoffel tensor.

by Einstein (The meaning of R.C. tensor)

Lowering the upper index β we get, Gauss

$$B_{\alpha\mu\nu\sigma} = g_{\alpha\beta} B_{\mu\nu\sigma}^\beta \quad \text{Curvature tensor,}$$

but This new associated tensor $B_{\alpha\mu\nu\sigma}$ is

called the Curvature Tensor (C.T)

+ The (C.T) $B_{\alpha\mu\nu\sigma}$ has the following

$$\text{properties: } B_{\alpha\mu\nu\sigma} = -B_{\mu\alpha\nu\sigma} \quad (i)$$

$$B_{\alpha\mu\nu\sigma} = -B_{\alpha\mu\sigma\nu} \quad (ii)$$

$$B_{\alpha\mu\nu\sigma} = B_{\nu\sigma\alpha\mu} \quad \text{has one symmetry (iii)}$$

(105) independent element)

$$B_{\mu\nu\sigma} + B_{\nu\sigma\mu} + B_{\sigma\mu\nu} = 0 \quad (iv)$$

These properties will reduce the number of independent components of $B_{\mu\nu\sigma}$ from 256 to 20 only.

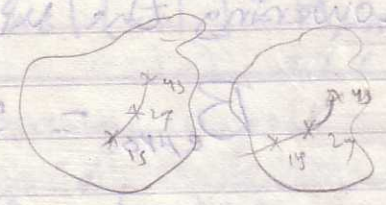
$B_{\mu\nu\sigma} = g_{\sigma\beta} B^{\beta}_{\mu\nu}$ and Gauss Curvature Tensor

$$B^{\beta}_{\mu\nu} = \left\{ \begin{matrix} \beta \\ \mu\sigma \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \beta \\ \mu\nu \end{matrix} \right\}_{,\sigma} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\sigma \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\nu \end{matrix} \right\}$$

What is Meant by an Invariant

$$I(x^{\mu}) = \bar{I}(\bar{x}^{\mu}) \text{ (in another system)}$$

which is a function of the coordinates



System A System B

$$\bar{x}^{\mu} = x^{\nu}(S) \text{ is a function}$$

$$B_{\mu\nu\sigma} = T^{\alpha}_{\mu} T^{\beta}_{\nu} T^{\gamma}_{\sigma} B_{\alpha\beta\gamma}$$

- (i) $B_{\mu\nu\sigma} = B_{\nu\sigma\mu} = B_{\sigma\mu\nu}$
- (ii) $B_{\mu\nu\sigma} + B_{\nu\sigma\mu} + B_{\sigma\mu\nu} = 0$
- (iii) $B_{\mu\nu\sigma} = g_{\sigma\beta} B^{\beta}_{\mu\nu}$

$B_{\mu\nu\epsilon}^\epsilon = -B_{\mu\epsilon\nu}^\epsilon \Rightarrow$ the contracted tensors $B_{\mu\epsilon\sigma}^\epsilon, B_{\mu\nu\epsilon}^\epsilon$ may become one contracted tensor only.
 notice, in $B_{\mu\epsilon\sigma}^\epsilon$ if σ became $\nu \Rightarrow B_{\mu\epsilon\sigma}^\epsilon = B_{\mu\epsilon\nu}^\epsilon = -B_{\mu\nu\epsilon}^\epsilon$. Hence one contracted tensor only.

(106)

3. Contracted Curvature Tensor: Ricci Tensor

The Riemann - Christoffel
 Contracted in 2 different ways:

tensor can be
 skew symmetric in (105)
 not 3 methods

i) by Setting $\mu = \epsilon$:

$$B_{\epsilon\nu\sigma}^\epsilon = \left\{ \begin{matrix} \epsilon \\ \epsilon\sigma \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \epsilon \\ \epsilon\nu \end{matrix} \right\}_{,\sigma} + \left\{ \begin{matrix} \alpha \\ \epsilon\sigma \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \epsilon\nu \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\sigma \end{matrix} \right\} = \boxed{\alpha, \epsilon \text{ dummy}}$$

Let α trace ϵ be σ

$$\frac{\partial^2 \log |g|}{\partial x^\sigma \partial x^\nu} - \frac{\partial^2 \log |g|}{\partial x^\nu \partial x^\sigma} + \left\{ \begin{matrix} \alpha \\ \epsilon\sigma \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \epsilon\nu \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\sigma \end{matrix} \right\}$$

= 0. This result was checked Einstein for it has enabled him to get rid of so many things.

(ii) By Setting $\sigma = \epsilon$

$$B_{\mu\nu\epsilon}^\epsilon = R_{\mu\nu} = \left\{ \begin{matrix} \epsilon \\ \mu\epsilon \end{matrix} \right\}_{,\nu} - \left\{ \begin{matrix} \epsilon \\ \mu\nu \end{matrix} \right\}_{,\epsilon} + \left\{ \begin{matrix} \alpha \\ \mu\epsilon \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\epsilon \end{matrix} \right\} =$$

ϵ dummy $\Rightarrow$ let ϵ be α

$$\frac{\partial^2 \log |g|}{\partial x^\mu \partial x^\nu} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}_{,\alpha} + \left\{ \begin{matrix} \alpha \\ \mu\epsilon \end{matrix} \right\} \left\{ \begin{matrix} \epsilon \\ \alpha\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{\partial \log |g|}{\partial x^\alpha}$$

(it value was found by computer)

α, ϵ dummy
let α be ϵ be β

$$R_{\mu\nu} = \frac{\partial^2 \log |g|}{\partial x^\mu \partial x^\nu} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{\partial \log |g|}{\partial x^\alpha} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}_{,\alpha} + \left\{ \begin{matrix} \beta \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \beta\nu \end{matrix} \right\}$$

" This is Relativity, after Newton's Gravitational Law was failed.
 Ricci Tensor "

$$R_{\mu\nu} = 0$$

(107)

مسار = path
مسار = path
مسار = path

Note: Einstein's Field Equation for

empty space are $R_{\mu\nu} = 0$

"its path is the one that is made as a result of the presence of the sun, source of gravity, after it has fallen on the space." on the



*4. Calculation of the Christoffel Symbols:

To Calculate Christoffel Symbols in the special case for Riemannian spaces whose metric has the diagonal form:

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{44}(dx^4)^2$$

$$g_{\mu\nu} = 0 \text{ if } \mu \neq \nu$$

$$g = \begin{vmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{vmatrix} = g_{11} g_{22} g_{33} g_{44}$$

Hence in this special case

$$g^{\mu\mu} = \frac{\text{Cof } g_{\mu\mu} \text{ in } g}{g} = 1/g_{\mu\mu} \Rightarrow$$

$$g^{\mu\nu} = 1/g_{\mu\nu} \quad ; \quad g^{\mu\mu} = 0 \quad \text{if } \mu \neq \nu$$

thus the non-vanishing members of the Christoffel Symbols are one of the following 3 forms.

(i) $\left\{ \begin{smallmatrix} \mu \\ \mu\nu \end{smallmatrix} \right\}$ Summation Convention is suspended!

i.e. No Summation: one Symbol only, $\mu \neq \nu$

$$\left\{ \begin{smallmatrix} \mu \\ \mu\nu \end{smallmatrix} \right\} = \frac{1}{2} g^{\mu\alpha} \left[\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right]$$

Remember $g^{\mu\alpha} = 0$ as long as $\mu \neq \alpha$

$$= \frac{1}{2} g^{\mu\nu} \left[\frac{\partial g_{\mu\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\nu} \right]$$

one term only when $\alpha = \nu$ which is in accordance with this special case of Riemannian spaces.

Since $\mu \neq \nu$

$$\left\{ \begin{smallmatrix} \mu \\ \mu\nu \end{smallmatrix} \right\} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\mu} = \frac{1}{2 g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial x^\mu} = \frac{\partial \log \sqrt{|g_{\mu\nu}|}}{\partial x^\mu}$$

which is our 1st result (1)

(ii) $\left\{ \begin{smallmatrix} \mu \\ \mu\mu \end{smallmatrix} \right\}$ one Symbol only (no summation).

$$\left\{ \begin{smallmatrix} \mu \\ \mu\mu \end{smallmatrix} \right\} = \frac{\partial \log \sqrt{|g_{\mu\mu}|}}{\partial x^\mu} \quad (2)$$

(109)

(iii)

$$\{^{\mu}_{\mu\mu}\}$$

if with conditions $\mu \neq \nu$, No summation

$$\{^{\nu}_{\mu\mu}\} = \frac{1}{2} g^{\nu\alpha} \left[\frac{\partial g_{\mu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\mu}}{\partial x^{\alpha}} \right]$$

one term only when $\alpha = \nu$

$$\frac{1}{2} g^{\nu\nu} \left[\frac{\partial g_{\mu\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\mu}}{\partial x^{\nu}} \right] =$$

$$\{^{\nu}_{\mu\mu}\} = - \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x^{\nu}} \quad \text{--- (3)}$$

(iv) $\{^{\alpha}_{\mu\nu}\} = 0$ if $\mu \neq \nu \neq \alpha$

$$\frac{1}{2} g^{\alpha\lambda} \left[\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\nu\mu}}{\partial x^{\lambda}} - \frac{\partial g_{\mu\mu}}{\partial x^{\lambda}} \right]$$

* ex.

Calculate the Christoffel.

3-index Symbols for the metric

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

(length (t=CT))

$$\nu = \nu(r), \quad \lambda = \lambda(r)$$

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = t$$

$$g_{11} = e^{2\nu}, \quad g_{22} = -r^2$$

$$g_{33} = -r^2 \sin^2\theta, \quad g_{44} = e^{2\nu}$$

$$g = \begin{vmatrix} -e^{2\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & e^{2\nu} \end{vmatrix} = -e^{2(\lambda+\nu)} r^4 \sin^2 \theta.$$

$$g^{11} = -e^{-2\lambda}$$

$$g^{33} = -1/r^2 \sin^2 \theta.$$

$$g^{22} = -1/r^2$$

$$g^{44} = e^{-2\nu}$$

$$\{1,1\} = \frac{\partial \log \sqrt{-g_{11}}}{\partial r} = \lambda'$$

الدرجة الأولى اننا نلاحظ اننا نلاحظ

في الحالة العامة

$$\{1,1\}, \{1,1\}, \{1,1\}$$

dead

$$\{1,2\}, \{1,2\} = \frac{\partial \log \sqrt{-g_{22}}}{\partial r} = 1/r.$$

$$\{1,2\}, \{1,3\} =$$

$$\{1,3\} = \frac{\partial \log \sqrt{-g_{33}}}{\partial r} = 1/r$$

$$\{2,2\} = -\frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial r} = -\frac{1}{2} (-e^{-2\lambda}) (-2r) = -r e^{2\lambda}.$$

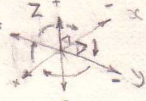
$$\{3,3\} = -\frac{1}{2} g^{22} \frac{\partial g_{33}}{\partial \theta} = -\frac{1}{2} \left(-\frac{1}{r^2}\right) (-r^2 \times 2 \sin \theta \times \cos \theta) =$$

$$= -\sin \theta \cos \theta$$

(111)

((floor = storey = a level of a building))

{ Z floor
y flats x flats }



((Right handed system))
((3rd quadrant))

if $\begin{pmatrix} u \\ v \end{pmatrix}$

1 1	2 1	3 1	4 1	$\partial \log \sqrt{e^{2u}} / \partial r = \frac{1}{r}$	$\partial \log \sqrt{e^{2u}} / \partial \theta = 0$	$\partial \log \sqrt{e^{2u}} / \partial \phi = 0$	$\partial \log \sqrt{e^{2u}} / \partial t = 0$
1 2	2 2	3 2	4 2	$\partial \log \sqrt{e^{2u}} / \partial \theta = 0$	$+\frac{1}{2} e^{2u} \frac{r^2}{r^2} = \frac{1}{2}$		
1 3	2 3	3 3	4 3	$\partial \log \sqrt{e^{2u}} / \partial \phi = 0$		$+\frac{1}{2} e^{2u} \frac{\partial \sin \theta}{\partial r} = -r \sin \theta \cdot e^{2u}$	$\frac{e^{2u}}{e \cdot e} \frac{d\theta}{dr} + \frac{1}{2} e^{2u} \frac{\partial^2 u}{\partial r^2} =$
1 4	2 4	3 4	4 4	$\partial \log \sqrt{e^{2u}} / \partial t = 0$			
2 1	2 2	3 1	4 1	$+\frac{1}{2} r^2 \partial - e^{2u} / \partial \theta = 0$	$\partial \log \sqrt{r^2} / \partial r = 1/r$		
2 2	2 2	3 2	4 2	$\partial \log \sqrt{r^2} / \partial r = 1/r$	$\partial \log \sqrt{r^2} / \partial \theta = 0$	$\partial \log \sqrt{r^2} / \partial \phi = 0$	$\partial \log \sqrt{r^2} / \partial t = 0$
2 3	2 3	3 3	4 3		$\partial \log \sqrt{r^2} / \partial \phi = 0$	$+\frac{1}{2} r^2 \frac{\partial \sin \theta}{\partial \theta} = -\sin \theta \cdot \cos \theta$	
2 4	2 4	3 4	4 4		$\partial \log \sqrt{r^2} / \partial t = 0$		$+\frac{1}{2} r^2 \partial e^{2u} / \partial t = 0$
3 1	3 1	3 1	4 1	$+\frac{1}{2} (r \sin \theta)^2 \partial - e^{2u} / \partial \phi = 0$		$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial r = 1/r$	
3 2	3 2	3 2	4 2		$+\frac{1}{2} (r \sin \theta)^2 \partial - r^2 / \partial \phi = 0$	$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial \theta = \cot \theta$	
3 3	3 3	3 3	4 3	$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial r = 1/r$	$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial \theta = \cot \theta$	$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial \phi = 0$	$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial t = 0$
3 4	3 4	3 4	4 4			$\partial \log \sqrt{r^2 \sin^2 \theta} / \partial t = 0$	$+\frac{1}{2} (r \sin \theta)^2 \partial e^{2u} / \partial \phi = 0$
4 1	4 1	4 1	4 1	$-\frac{1}{2} e^{2u} \partial - e^{2u} / \partial t = 0$			$\partial \log \sqrt{e^{2u}} / \partial r = 1/r$
4 2	4 2	4 2	4 2		$-\frac{1}{2} e^{2u} \partial - r^2 / \partial t = 0$		$\partial \log \sqrt{e^{2u}} / \partial \theta = 0$
4 3	4 3	4 3	4 3			$-\frac{1}{2} e^{2u} \partial - r^2 \sin \theta / \partial t = 0$	$\partial \log \sqrt{e^{2u}} / \partial \phi = 0$
4 4	4 4	4 4	4 4	$\partial \log \sqrt{e^{2u}} / \partial r = 1/r$	$\partial \log \sqrt{e^{2u}} / \partial \theta = 0$	$\partial \log \sqrt{e^{2u}} / \partial \phi = 0$	$\partial \log \sqrt{e^{2u}} / \partial t = 0$

13 only survived

(Living)

(born dead)

(not able to give birth)

$$= (1.5 - 1) \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

$$= (2.0 - 1.5) \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

(relationship between Phy and Geometry appears in field equations)

$$R_{\mu\nu} = 0 \equiv \text{matter as the source of curvature of space} \equiv \text{R.H.S}$$

space
matter

(112)

General Theory of Relativity

(Mach philosophy)

Matter Source = matter sink

and

Space (Source) that causes curvature is affected by source

hence matter ~ curvature
space ~ curvature

Curvature starts it

brain's memory tracks

Procedural Alg

1. Geometrization of Physics

In his general theory of relativity Einstein assumes that a 4-dimensional Riemannian Space gives us a good picture of all the physical world around us with all its energetic and material contents.

He has chosen for his field equations for the empty space surrounding any material distribution,

$$R_{\mu\nu} = 0 \quad \text{Ricci Tensor}$$

material distribution = Sum

Vacuum = empty space = matter = 0

The R.H.S is the 0 denotes the matter present in the region surrounding the material distribution; and because this region is Vacuum, thus matter present = 0

$$\frac{\partial^2 \log \sqrt{-g}}{\partial x^\mu \partial x^\nu} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{\partial \log \sqrt{-g}}{\partial x^\alpha} - \left\{ \begin{matrix} \alpha \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \nu\alpha \end{matrix} \right\} = 0$$

(Only thing put in this Vacuum Curves it)

"10 equations"

Further sheet
not
putting on it
(1112) (1112) (1112)

$$R_{\mu\nu} = \frac{\partial^2 \log \sqrt{-g}}{\partial x^\mu \partial x^\nu} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \frac{\partial \log \sqrt{-g}}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \nu\beta \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\alpha \end{matrix} \right\} \frac{\partial \log \sqrt{-g}}{\partial x^\nu} = 0$$

We also assumed that the trajectory of a free test particle placed in the field is a geodesic

$$\frac{d^2 x^\sigma}{ds^2} + \left\{ \begin{matrix} \sigma \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

for light $\Rightarrow$ Photon
its geodesic

And the path of a ray of light is a null-geodesic $ds = 0$ along the (n-geodesic)

Analogies from Classical Mechanics can be used to convince

slightly, people. $B_{\mu\nu}^e = \left\{ \begin{matrix} e \\ \mu\nu \end{matrix} \right\} - \left\{ \begin{matrix} e \\ \mu\nu \end{matrix} \right\}_0 + \left\{ \begin{matrix} \alpha \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} e \\ \nu\alpha \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} e \\ \nu\alpha \end{matrix} \right\}$

$$B_{\mu\nu}^e = B_{\mu\nu}^e = R_{\mu\nu}$$

In order to satisfy Einstein Field eqn. $R_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \nu\alpha \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \nu\alpha \end{matrix} \right\} + \left\{ \begin{matrix} e \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \nu\alpha \end{matrix} \right\} - \left\{ \begin{matrix} e \\ \mu\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \nu\alpha \end{matrix} \right\} = 0$

Most general space curved
(113)

2. The Schwarzschild's Exterior Solution

(the thing put on the rubber sheet)

To find out the field of a gravitating particle at rest at the origin.

The 4-dimensional Riemannian Space representing it is assumed to possess spherical symmetry.

The simplest form of metric having spherical symmetry is taken to be of the form,

The metric of Special Relativity

(The Galilean frame) is given by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad (\text{Sp. Rel})$$

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \quad (\text{Sp. Rel})$$

which is a flat space $\rightarrow$ wanted curved space.

(use functions)

Hence the metric used by Schwarzschild was

$$ds^2 = e^{\lambda} dt^2 - e^{\nu} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

$$\nu = \nu(r); \quad \lambda = \lambda(r)$$

1st We find Christoffel symbols, also find $g_{\mu\nu}$

$$g_{\mu\nu} = 0 \quad \text{if } \mu \neq \nu$$

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = t$$

$\equiv$ CT length

$$\{a\} \{a\} - \{a\} \{a\} + \{a\} - \{a\} = 0 \quad (114)$$

$$g = \begin{vmatrix} 2\lambda & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & e^{2\nu} \end{vmatrix} = -e^{2\nu+2\lambda} r^4 \sin^2 \theta$$

$$\sqrt{-g} = e^{\nu+\lambda} r^2 \sin \theta$$

$$\log \sqrt{-g} = \nu + \lambda + 2 \log r + \log \sin \theta$$

$$\frac{\partial \log \sqrt{-g}}{\partial r} = \nu' + \lambda' + \frac{2}{r}$$

$$\text{The DE of a geodesic is: } \frac{d^2 x^\alpha}{ds^2} + \left\{ \begin{matrix} \alpha \\ \mu \nu \end{matrix} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

The non-vanishing elements of the

Christoffel Symbols are one or other

$$\text{of the forms (i) } \left\{ \begin{matrix} \nu \\ \mu \nu \end{matrix} \right\} ; \mu \neq \nu ; = \frac{\partial \sqrt{-g_{\mu\nu}}}{\partial x^\mu}$$

$$(ii) \left\{ \begin{matrix} \mu \mu \\ \mu \mu \end{matrix} \right\} = \frac{\partial \log \sqrt{-g_{\mu\mu}}}{\partial x^\mu} ; \mu = \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x^\mu} = \left\{ \begin{matrix} \mu \\ \mu \mu \end{matrix} \right\}$$

Hence we get the following table for the

non-vanishing Symbols

x is a function that describes the flow (like a fluid) of the quantity x in all spaces i.e. in any region of the space whatever small region or large region is as long as the quantity (x) does not die i.e. $x \neq 0$, there exist a flow in any space depending on x itself.

115 Ricci Tensor: $R_{\mu\nu} = \{\alpha_{\mu\alpha}\}_{,\nu} - \{\alpha_{\mu\nu}\}_{,\alpha} + \{\beta_{\mu\alpha}\}\{\alpha_{\beta\nu}\} - \{\beta_{\mu\nu}\}\{\alpha_{\beta\alpha}\}$

$\{1,1\} = 2$ $\{2,2\} = 1/r$

$\{1,2\} = 1/r$

$\{2,2\} = re^{-2\lambda}$

$\{1,3\} = 1/r$

$\{2,3\} = \cot\theta$

$\{1,4\} = v'$

$\{3,1\} = 1/r$

$\{4,1\} = v'$

$\{3,2\} = \cot\theta$

$\{4,2\} = e^{2\lambda-2\psi} v'$

$\{3,3\} = -r \sin^2\theta e^{-2\lambda}$

$\{3,3\} = -\sin\theta \cos\theta$

For a diagonal metric space
 $\{\mu_{\mu\mu}\} = \partial \log \sqrt{-g} / \partial x^\mu, \mu \neq \nu$
 $\{\mu_{\mu\mu}\} = \partial \log \sqrt{-g} / \partial x^\mu$
 $\{\mu_{\nu\nu}\} = -1/2 g^{\mu\mu} (\partial g^{\nu\nu} / \partial x^\mu), \mu \neq \nu$
 $\{\mu_{\nu\nu}\} = 0, \mu \neq \nu \neq 0, g_{\mu\nu} = 0, \mu \neq \nu.$

2nd Then Substituting into the field equations for empty space:

$R_{\mu\nu} = \partial \log \sqrt{-g} / \partial x^\mu \partial x^\nu - \{\alpha_{\mu\alpha}\}_{,\nu} + \{\beta_{\mu\alpha}\}\{\alpha_{\beta\nu}\} - \{\alpha_{\mu\nu}\}\{\beta_{\beta\alpha}\} = 0$

$R_{\mu\nu} = 0$

we get:

$R_{11} = \left[\frac{\partial^2 \log \sqrt{-g}}{\partial r^2} - \{1,1\} \frac{\partial \log \sqrt{-g}}{\partial x^1} \right]$ (Go To the table You Constructed)

$= \{1,1\}_{,\alpha} + \{\beta_{1\alpha}\}\{\alpha_{\beta 1}\} = \left(v'' + \lambda'' - \frac{2}{r^2} \right)$

$= \{1,1\} \frac{\partial \log \sqrt{-g}}{\partial r} - \frac{\partial \{1,1\}}{\partial r} +$

$$+ \left[\{1,1\} \{1,1\} + \{2,2\} \{2,2\} + \{3,3\}^2 + \{4,4\}^2 \right] \}$$

$$R_{11} = \left(v'' + \lambda'' - \frac{2}{r^2} \right) - \lambda' \left(v' + \lambda' + \frac{2}{r} \right) - \lambda'' + \lambda'^2 + 1/r^2 + 1/r^2 + v'^2 =$$

$$v'' + v'^2 - \lambda' v' - 2\lambda'/r$$

Verify Homeworks

$$v'' + v'^2 - \lambda' v' - 2\lambda'/r = 0 \quad (1)$$

$$R_{22} = e^{2\lambda} [r(v' + \lambda') + 1] - 1 = 0 \quad (2)$$

$$R_{33} = \sin^2 \theta R_{22} \quad (3)$$

$$R_{44} = e^{2v-2\lambda} \left[-v'' - v'^2 + \lambda' v' - \frac{2v'}{r} \right] = 0 \quad (4)$$

3 DE's

which are the field equations as easily solved DE's

From (1), (3) $\frac{2\lambda'}{r} + \frac{2v'}{r} = 0$

$$\lambda' + v' = 0 \quad (4)$$

Integrating $\Rightarrow$ we get

$$v = -\lambda + \text{Const.}$$

Physics $\rightarrow$

$$(5)$$

rubber sheet: (as curvature $\rightarrow$) $\Rightarrow$ (rubber sheet)
 flat at infinity.

(117)

But since the field is flat when r

r tends to infinity hence

$\lambda \rightarrow 0$, $\lambda \rightarrow 0$ as $r \rightarrow \infty$

λ Constant $= 0$

((remember View from inside and))

((View from outside)) (i)

Substituting from (i) into (3) we get

$$e^{2\lambda} [-2r\lambda' + 1] = 1 \quad \text{which is}$$

$$\frac{d}{dr} [r e^{-2\lambda}] = 1$$

Integrating $\Rightarrow r = r e^{-2\lambda} + C$

$C = -2m$

[which may denote that what we may view as mass from inside is no more than a distance approximation from outside. mass is the body emerged as the result of abstract space that shaped the structure. hence distance]

$$e^{-2\lambda} = 1 - \frac{2m}{r}$$

$$e^{2\lambda} = e^{-2\lambda} = 1 - \frac{2m}{r}$$

Hence we get the final solution with the

metric

(the good name)

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - 2m/r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad [1]$$

Which is known as Schwarzschild's exterior

Solution .

Note that the constant "m" is so far just a constant of integration.

All this is geometry

where is physics?

2nd

3. Newton's Equations of Motion as a

First Approximation of General Relativity:

(relation between classical units and relativistic units)

Relativistic units :

To Compare the general Theory of relativity with the Newtonian classical mechanics ,

1st We must consider a very weak, static field

whose metric differs slightly from that the flat space of special relativity . i.e. from the Galilean frame

ds^2

(119)

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad (1)$$

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (2)$$

$$g_{11} \approx -1, \quad g_{22} \approx -1 \quad (3)$$

$$g_{33} \approx -1, \quad g_{44} \approx 1$$

2nd ~~test~~ ~~particles~~ ~~whose~~ very slow test particles whose velocity given by:

$$\frac{dx^1}{ds} \approx 0, \quad \frac{dx^2}{ds} \approx 0$$

$$\frac{dx^3}{ds} \approx 0$$

Hence from (2)

$$\text{we get } \frac{dx^4}{ds} \approx 1$$

The trajectory of any test particle is the

Geodesic

$$\frac{d^2 x^\sigma}{ds^2} + \{ \sigma \mu \nu \} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

$$\frac{d^2 x^0}{ds^2} + \{ \sigma \mu \nu \} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

$$+ \{ 44 \} 1 \times 1 = 0$$

(4)

$$\text{But } \{ 44 \} = -\frac{1}{2} g^{\sigma\sigma} \frac{\partial g_{44}}{\partial x^\sigma}$$

Since the field is static i.e. independent of time,

$$g'' = \lambda / g'' = -1 \quad \text{at } g'' = 0 \text{ and } g'' = -1$$

$$\Rightarrow \left\{ \begin{matrix} 0 \\ 4 \end{matrix} \right\} = -\frac{1}{2}(-1) \frac{\partial g_{44}}{\partial x^0} = \frac{1}{2} \frac{\partial g_{44}}{\partial x^0}, \quad \sigma = 1, 2, 3$$

$$\frac{d^2(x^0)}{ds^2} = -\frac{1}{2} \frac{\partial g_{44}}{\partial x^0} \quad \text{--- (5)}$$

3rd Then introducing the new function ψ , defined by

$$\psi = \frac{c^2(g_{44} - 1)}{2} \quad \text{--- (7)}$$

Hence eqn (6), noting that $\frac{dx^4}{ds} \approx 1$

$$c dt \approx ds \Rightarrow c^2 dt^2 \approx ds^2 \Rightarrow \frac{d^2 x^0}{ds^2} \text{ becomes}$$

$$\therefore \frac{1}{c^2} \frac{d^2 x^0}{dt^2} = -\frac{1}{2} \frac{\partial g_{44}}{\partial x^0} = -\frac{1}{c^2} \frac{\partial \psi}{\partial x^0}$$

$\rightarrow$ From (7)

$$\frac{d^2 x^0}{ds^2} = -\frac{\partial \psi}{\partial x^0} \quad \sigma = 1, 2, 3$$

$$\sigma = 1 \quad \frac{d^2 x}{dt^2} = -\frac{\partial \psi}{\partial x}$$

$$\sigma = 2 \quad \frac{d^2 y}{dt^2} = -\frac{\partial \psi}{\partial y}$$

$$\sigma = 3 \quad \frac{d^2 z}{dt^2} = -\frac{\partial \psi}{\partial z}$$

$$\sigma = 4$$

(122)

Which are exactly the Newtonian / Classical

equations for a field whose potential is given by

$$\psi = \frac{c^2 (g_{44} - 1)}{2}$$

4th According to this study we can find out the Physical Meaning of the Schwarzschild exterior

Solution

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - dr^2 / \left(1 - \frac{2m}{r}\right) - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$g_{44} = 1 - \frac{2m}{r} \Rightarrow \frac{c^2}{2} g_{44} = \frac{c^2}{2} - \frac{mc^2}{r} \Rightarrow -\frac{mc^2}{r} = \frac{c^2}{2} (g_{44} - 1)$$

$$\psi = \frac{c^2}{2} \left(1 - \frac{2m}{r}\right)$$

if $m = \frac{1}{2} r \Rightarrow$
 g_{44} becomes zero

$$\psi = - \frac{mc^2}{r} \quad \text{--- (i)}$$

But in classical mechanics M

The Potential due to a particle a particle of mass M (grams) at distance r (cm) is given by

$$\psi = - \frac{\gamma M}{r} \quad \text{--- (ii)}$$

If the two potentials (i), (ii) are identical

$$mc^2 = \gamma M$$

length

constants

mass

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r^2} \Rightarrow F = G \frac{m_1 m_2}{r^2} \Rightarrow \frac{\text{Force}}{(\text{mass})^2} \cdot (\text{distance})^2 = \frac{\text{dyne} \cdot \text{cm}^2}{\text{gm}^2}$$

$$\text{dyne} = \text{gm} \cdot \frac{\text{cm}}{\text{sec}^2} \Rightarrow G = \gamma = \frac{\text{cm}^3}{\text{gm} \cdot \text{sec}^2} = \frac{\text{cm}}{\text{gm}} \cdot \left(\frac{\text{cm}}{\text{sec}}\right)^2$$

(123)

$$m = \frac{\gamma M}{c^2}$$

" γ Newton's gravitational Constant "

For example: In the case of the Sun

$$M = 1.99 \times 10^{33} \text{ gram} \approx 2 \times 10^{33} \text{ gr}$$

$$\gamma = 6.664 \times 10^{-8} \left(\frac{\text{cm}}{\text{gm}}\right) \cdot \left(\frac{\text{cm}}{\text{sec}}\right)^2$$

$$c = 3 \times 10^{10} \text{ cm/sec}$$

$$m = \frac{\gamma M}{c^2} \approx 1.47 \text{ Km}$$

Note that the relativistic Constant m , representing the mass of the particle is in "cm"

"the relativistic Constant m ; as a quantity in the L.H.S facing the quantity in the R.H.S, the mass M of the particle, is in cm."

This is to be expected:

$\theta = \text{sr} = \text{pure number}$
dimensionless.

Relativistic heavy side units

$$m = \gamma M / c^2 \quad \text{cm}$$

$$t = cT \quad \text{cm}$$

$$r = R \quad \text{cm}$$

Space-time (T)

(124)

Planetary Orbits:

According to the Schwar

Space time

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

the trajectory of a free test-particle is the geodesic

$$\frac{d^2 x^\mu}{ds^2} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \quad (\text{Four ones})$$

$$\sigma=1 \quad \therefore \frac{d^2 r}{ds^2} + \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} \left(\frac{dr}{ds}\right)^2 + \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \left(\frac{d\theta}{ds}\right)^2 + \left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} \left(\frac{d\phi}{ds}\right)^2 + \left\{ \begin{matrix} 1 \\ 44 \end{matrix} \right\} \left(\frac{dt}{ds}\right)^2 = 0$$

$\sigma=2$

$$\frac{d^2 \theta}{ds^2} + 2 \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} \frac{dr}{ds} \cdot \frac{d\theta}{ds} + \left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} \left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2 \theta}{ds^2} + \frac{2}{r} \frac{dr}{ds} \cdot \frac{d\theta}{ds} - \sin\theta \cos\theta \left(\frac{d\phi}{ds}\right)^2 = 0 \quad (ii)$$

$\sigma=3$

$$\frac{d^2 \phi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \cdot \frac{d\phi}{ds} + 2 \cot\theta \frac{d\theta}{ds} \frac{d\phi}{ds} = 0 \quad (iii)$$

$\sigma=4$

$$\frac{d^2 t}{ds^2} + 2 \frac{dr}{ds} \cdot \frac{dt}{ds} = 0 \quad (iv)$$

Which are the 4 DE'S of the geodesic.

If initially the test particle is placed at rest in the plane $\theta = \pi/2$

$$d\theta/ds = 0, \quad \theta = \pi/2 \quad (\text{initial condition})$$

*) (0 = $\pi/2$ is a circle in the plane $\theta = \pi/2$)

Description of motion $\rightarrow$ Kinematics is $\underline{v}, \underline{a}$

Causes of motion $\rightarrow$ what is $\underline{F}$ & $\underline{g}$ (gravitational)

$$\Rightarrow \text{if } \underline{v}|_{t=0} = 0, \quad \underline{a}|_{t=0} = 0$$

(if $\underline{v}, \underline{a} = 0$)

G. Rel. General Relativity

therefore we have to insure that $\frac{d^2\phi}{ds^2} = 0$

But from (ii) we get, initially

$$\frac{d^2\phi}{ds^2} + 0 = 0 \Rightarrow \frac{d^2\phi}{ds^2} = 0$$

$V_0 = 20, a_0 = 0$

Therefore, the particle will continue to move in the plane $\theta = \pi/2$.

From (iii) we get $\frac{d^2\phi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \cdot \frac{d\phi}{ds} = 0$

let $v = d\phi/ds \Rightarrow \frac{1}{v} \frac{dv}{ds} + \frac{2}{r} \frac{dr}{ds} = 0$

Integrating w.r to s we get
 $\log v + 2 \log r = \text{const}$

$$r^2 \frac{d\phi}{ds} = h \quad (I)$$

Also from (iv) taking $\omega = \frac{dt}{ds}$

$$\frac{1}{\omega} \frac{d\omega}{ds} = -2 \frac{dV}{dr} \cdot \frac{dr}{ds} = -2 \frac{dV}{ds}$$

Integrating w.r to s we get

$$\log \omega = -2V + \text{const}$$

$$\frac{dt}{ds} = k e^{2V} = k [1 - (2m/r)] \quad (II)$$

To get $\frac{dr}{ds}$ we substitute from (I), (II) into the Space-time **[1]**

(12.6) $\phi \equiv$ Newtonian potential Φ

$$1 = \left(1 - \frac{2m}{r}\right) \left(\frac{dt}{ds}\right)^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)} \left(\frac{dr}{ds}\right)^2 - r^2 \left(\frac{d\phi}{ds}\right)^2 \quad (I)$$

$$1 = \left(1 - \frac{2m}{r}\right) \left(\frac{dt}{ds}\right)^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)} \left(\frac{dr}{ds}\right)^2 - r^2 \frac{h^2}{r^4}$$

$$1 = \left(1 - \frac{2m}{r}\right) \frac{k^2}{\left(1 - \frac{2m}{r}\right)^2} - \frac{1}{\left(1 - \frac{2m}{r}\right)} \left(\frac{dr}{ds}\right)^2 - r^2 \frac{h^2}{r^4}$$

$$\left(\frac{dr}{ds}\right)^2 = k^2 - \left(1 - \frac{2m}{r}\right) - \frac{h^2}{r^2} \left(1 - \frac{2m}{r}\right)$$

$$\left(\frac{dr}{ds}\right)^2 = (k^2 - 1) + \frac{2m}{r} - \frac{h^2}{r^2} \left(1 - \frac{2m}{r}\right) \quad (II)$$

We compare with Newton's $\Rightarrow$ get acceleration
we omit the time (t) which is in comparison with
measured implicitly $\frac{s}{r} + \frac{r}{2b} \frac{1}{r} \Rightarrow$ radial orbits

We then introduce the new variable

$$u = 1/r \quad \Rightarrow \quad h^2 u^2 = \frac{d\phi}{ds} \quad (I)$$

$$\frac{dr}{ds} = \frac{dr}{du} \cdot \frac{du}{d\phi} \cdot \frac{d\phi}{ds} =$$

$$= \frac{1}{u^2} \left(\frac{du}{d\phi}\right) h^2 u^2 = -h \left(\frac{du}{d\phi}\right) \quad \left| \frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2} \text{ Newton's } (DE \text{ of orbits}) \right.$$

hence from (II), we get:

which is a (u, ϕ) direct relation in between

which is what we want, to go

$$h^2 \left(\frac{du}{d\phi}\right)^2 = k^2 - 1 + 2mu - h^2 u^2 (1 - 2mu)$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{k^2 - 1}{h^2} + \frac{2m}{h^2} u - u^2 + 2m u^3$$

If $v^2 = h^2 \left(\frac{dx}{dt}\right)^2$, $\Rightarrow v^2 \frac{dx}{dt}$, $\Rightarrow \frac{d^2 x}{dt^2} = \ddot{x} = v \frac{dv}{dx}$

$$\left(\frac{d}{dx} (h^2 v^2)\right)$$

good one

I think it is μu^2

$$F = h^2 u^2 \left[\frac{d^2 u}{d\phi^2} + u \right]$$

$$\Rightarrow \left[\frac{d^2 u}{d\phi^2} + u \right] = \frac{F}{h^2 u^2} = \frac{m u^2}{h^2 u^2}$$

where
 $F \propto u^2 \Rightarrow$
 $F = \mu u^2$

act (X) $\alpha 1/2$ and diff. eq. w.r to u substituted in eq. of J1.

$$\frac{d^2 u}{d\phi^2} = -\frac{m}{h^2}$$

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2$$

Hence we get the diff. eqn for the planetary orbits

in the Schw

space-time into the form

$$r^2 \frac{d\phi}{ds} = h$$

(I)

cf (compare) Newton's Mech.

$$r^2 \frac{d\phi}{dt} = h$$

$$\frac{dt}{ds} = K/(1 - (2M/r))$$

(II)

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2$$

Central force $F = -\frac{h^2}{m} \left[\frac{d^2 u}{d\phi^2} + u \right]$

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2$$

mass $\frac{10^{-36}}{10^{-32}}$

Fahmy's experience

Ramsey Dynamics, Part 1

to get $3mu^2$

Einstein

DEs

by Forthyth. best one

Wess

The general solution of these diff. eqns involve elliptic function, and therefore it is not convenient for practical applications

It was obtained by Forthyth in 1934

$\frac{d^2 u}{d\phi^2}$
circle

$\frac{d^2 u}{d\phi^2}$
hyperbola

cm
in
pm

(128)

It is useful to solve these eqns by successive approximations we can get the 4 crucial (vital) tests:

(i) The advance of Perihelion of Mercury.

(ii) Gravitational deflection for a ray of light.

(iii) Red-shift of Spectral Lines.

In the 50's Satellites Laser reflects pulses

(iv) Radar echo - delay

(Experimental evidence of Relativity. (best))

1. The Advance of perihelion of Mercury

1st Approximation, we have

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2$$

$u = 1/r$

neglecting the 2nd order term in u i.e. $3mu^2$ we get

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{h^2}$$

Its solution is the Newtonian solution, i.e. the ellipse

$$u = \frac{m}{h^2} [1 + e \cos(\phi - \alpha)] \quad \text{--- (1)}$$

with 2 arbitrary constants

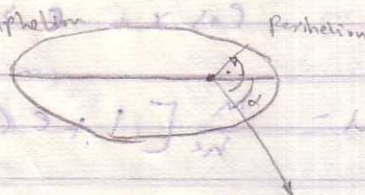
aphelion

peri = advance

2nd Approx.

Substituting from (1) into

the neglected term we get



$$\frac{d^2 u}{d\varphi^2} + u = \frac{m}{h^2} + 3m \times \frac{m^2}{h^4} [1 + e \cos(\varphi - \alpha)]^2 =$$

$$\frac{m}{h^2} + \frac{3m^3}{h^4} [1 + 2e \cos(\varphi - \alpha) + e^2 \cos^2(\varphi - \alpha)]$$

Note that the only term on the R.H.S. which will give an effect in the answer is the term in $\cos(\varphi - \alpha)$.

The Particular Integral (P.I.) corresponding to this term is given by:

$$= u_1 = \frac{1}{D^2 + 1} (-A \cos \varphi)$$

where $A = \text{const} = 6 m^3 e / h^4$

$$u_1 = A \frac{1}{(D+i)(D-i)} R e^{i\varphi} = AR \frac{1}{2i} e^{i\varphi} \frac{1}{(D+i-i)}$$

$$AR \frac{e^{i\varphi}}{2i} \varphi = \frac{1}{2} A \varphi \sin \varphi =$$

$$\frac{3m^3 e}{h^4} \varphi \sin(\varphi - \alpha)$$

Hence the solution in the 2nd approx. will be

$$u = \frac{m}{h^2} [1 + e \cos(\varphi - \alpha)] + \frac{3m^3}{h^4} e \varphi \sin(\varphi - \alpha)$$

$$= \frac{m}{h^2} [1 + e \{ \cos(\varphi - \alpha) + \frac{3m^3}{h^2} \varphi \sin(\varphi - \alpha) \}]$$

(130)

$$\cos(x + e) \approx \sum_{n=0}^{\infty} \frac{(x+e)^{2n}}{2n!} = \cos x \cos e + \sin x \sin e$$

$$\cos x + e \sin x$$

$$u = \frac{m}{h^2} [1 + e \cos(\varphi - \alpha)] \quad \text{which is the 2nd Approx.}$$

$$\delta \alpha \approx \frac{3m^2}{h^2} \varphi$$

Since the Newtonian solution is

$$u = \frac{m}{h^2} [1 + e \cos(\varphi - \alpha)]$$

which represents an ellipse with

α longitude of the perihelion

$$\text{Thus, Advance in perihelion} = \delta \alpha \approx \frac{3m^2}{h^2} \varphi$$

$$\frac{3m^2}{h^2} \times (2\pi) \quad \text{((every revolution))}$$

$$= \frac{6\pi m^2}{h^2}$$

$$\text{Astronomers use it} = \frac{6\pi m}{a(1-e^2)}$$

(a = major axis)

For Mercury advance = $43''$ seconds of arc per observation
every Century (100 revolutions of Mercury)

In 1845, Le Verrier $42.9''$
(For calculation)

For Venus $8.6''$

Earth $3.8''$

$1'' = 60 \text{ min}$
 $1 \text{ min} = 60 \text{ sec}$

2. Gravitational Deflection of a ray of light.

For a ray of light, is along the path of a ray of light $ds = 0$ (null geodesic)



But from the 1st integral (I) we have

$$r^2 \frac{d\phi}{ds} = h$$

Therefore for a ray of light, $h \rightarrow \infty$

Therefore the DE giving the path of a ray of light will be

From (III)
$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{r^2} + 3mu^2$$

Hence, For this case as $h \rightarrow \infty \Rightarrow$

$$\frac{d^2 u}{d\phi^2} + u = 3mu^2$$

* By 1st approximation:

$$\frac{d^2 u}{d\phi^2} + u = 0 \quad \text{i.e.} \quad \frac{d^2 u}{d\phi^2} = -u$$

Thus the solution for the 1st approxi can be taken

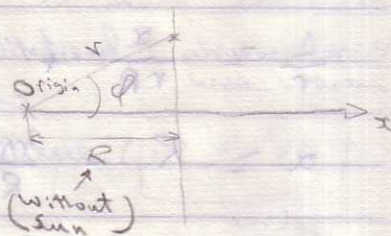
to be
$$u = \frac{1}{R} \cos \phi \quad \text{--- (3)}$$

which represents the straight line

$$x = R$$

$$\cos \phi = \frac{x}{R}$$

$$\frac{x}{R} = \frac{R}{r} \quad \therefore x = R$$



* 2nd approximation

Substi from (3) into the terms of the 2nd order

we get
$$(D^2 + 1)u = 3m \frac{\cos^3 \phi}{R^2} \quad \text{--- (4)}$$

The Particular Integral corresponding to R.H.S. is given by

(132)

$$u_1 = \frac{1}{D^2 + 1} \times \frac{3m}{R^2} \cdot \frac{1}{2} (1 + \cos 2\varphi)$$

$$u_1 = \frac{3m}{R^2} \frac{1}{D^2 + 1} (\cos^2 \varphi + \cos 2\varphi)$$

$$\frac{3m}{R^2} \left[1 + \frac{1}{(-4)^2 + 1} \cos 2\varphi \right] = \frac{3m}{2R^2} \left[1 - \frac{1}{3} \cos 2\varphi \right]$$
$$= \frac{3m}{3 \times 2 R^2} [3 - \cos 2\varphi] = \frac{m}{2R^2} [3 \cos^2 \varphi + 3 \sin^2 \varphi - (\cos^2 \varphi - \sin^2 \varphi)]$$

$$= \frac{m}{2R^2} (2 \cos^2 \varphi + 4 \sin^2 \varphi)$$

$$u_1 = \frac{m}{R^2} (\cos^2 \varphi + 2 \sin^2 \varphi) \quad \text{--- (6)}$$

Hence the solution in the 2nd approx is given by.

$$u = \frac{\cos \varphi}{R} + \frac{m}{R^2} (\cos^2 \varphi + 2 \sin^2 \varphi) \quad \text{--- (7)}$$

Using the Cartesian Coordinates

$x = r \cos \varphi$, $y = r \sin \varphi$, we get

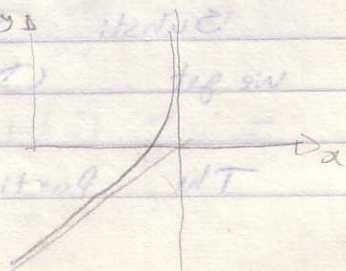
$$\frac{1}{r} = \frac{x}{rR} + \frac{m}{R^2} \left(\frac{x^2}{r^2} + 2 \frac{y^2}{r^2} \right)$$

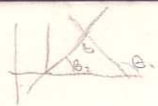
$$\boxed{x = R} \quad \text{--- (line)} \quad \frac{m}{R} \frac{2y^2 + x^2}{\sqrt{x^2 + y^2}} \quad \text{--- (curve)} \quad \text{asymptote} \quad \text{--- (8)}$$

The Def of the asymptote is the Curve that touches the barrier at infinity.

To find out the eqns of the two

asymptote (8), let $y \gg x$





$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$$

(133)

$$\Rightarrow x = R - \frac{m}{R} \frac{y^2(2 + \frac{x^2}{y^2})}{\sqrt{y^2(1 + \frac{x^2}{y^2})}}$$

$$x = R - \frac{m}{R} \frac{2y^2}{\pm y} = R \mp \frac{2m}{R} y \quad \text{--- (9)}$$

(9) gives the eqn of the two asymptotes with slopes.

$$m_1 = \frac{2m}{R}, \quad m_2 = -\frac{2m}{R}$$

Hence the angle between them (θ) is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{2m}{R} + \frac{2m}{R}}{1 - \frac{4m^2}{R^2}} \approx \frac{4m}{R}$$

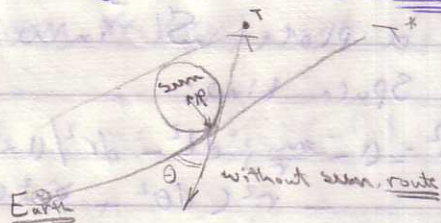
$$\theta = \frac{4m}{R} = \frac{4GM}{c^2 R}$$

For a ray of light just grazing the sun's Rim

$$R = 697,000 \text{ Kms}$$

$$m = 1.47 \text{ Kms}$$

$$\therefore \theta \approx 1.75 \text{ seconds of arc}$$



Note: Using Newton's gravitational Law and treating light as Corpuscles we get $\frac{1}{2}$ this value!

observe star hidden by the sun

"Prediction (deduction)"

This Result Can be verified by observing stars which are supposed to be hidden by the sun

(134)

Sun totally hidden

This can only be done during a total eclipse of the Sun

This was done:

(1) In 1919 in Brazil $1''.98$
In west Africa $1''.61$ (British)

(2) In 1922 (Americans)

$1''.72$, $1''.82$

(3) In 1952 $1''.70$

3. Gravitational Red-Shift of Spectral Lines:

+ Consider a vibrating atom stationary at a place S in the Schwarz.

Space-time:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

Since the atom is approx. stationary

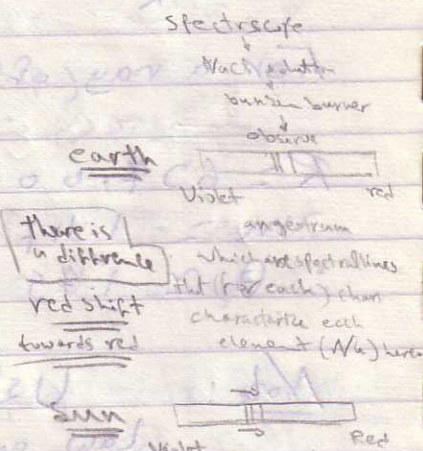
therefore, $dr=0$, $d\theta=0$, $d\phi=0$

$$\therefore ds^2 = \left(1 - \frac{2m}{r_s}\right) dt_s^2 \quad (2)$$

+ Consider another "exactly similar atom" vibrating at another place E in Space-time (1).

$$\therefore ds^2 = \left(1 - \frac{2m}{r_E}\right) dt_E^2 \quad (3)$$

The invariant (3) $\sim (ds)^2 = (2) \sim (ds)^2 \sim (4)$



The similarity of the two atoms is justified by taking ds to be the same for the two atoms.

$$\left(1 - \frac{2m}{r_s}\right) dt_s^2 = \left(1 - \frac{2m}{r_E}\right) dt_E^2$$

$$\therefore \frac{dt_s}{dt_E} = \frac{\sqrt{1 - (2m/r_E)}}{\sqrt{1 - (2m/r_s)}} \sim \left(1 - \frac{m}{r_E}\right) \left(1 + \frac{m}{r_s}\right)$$

$$\sim 1 + \frac{m}{r_s} - \frac{m}{r_E}$$

If S & E refer to the surface of the Sun and Earth, respectively, we can neglect m/r_s as compared with m/r_s .

$$\therefore \frac{dt_s}{dt_E} \sim 1 + \frac{m}{r_s} \quad (4)$$

If λ_s, λ_E are the wave-lengths of the vibration at S & E , respectively, ~~xx~~ λ_E , as

Compared with m/r_s

$$\therefore \frac{\lambda_s}{\lambda_E} = \frac{dt_s}{dt_E} \sim 1 + \frac{m}{r_s} \quad (5)$$

$$\text{Taking } \lambda_s = \lambda_E + \delta\lambda$$

$$\therefore \frac{\lambda_s}{\lambda_E} = 1 + \frac{\delta\lambda}{\lambda_E} \quad (6)$$

From (5), (6) we get

$$\frac{\delta\lambda}{\lambda_E} = \frac{m}{r_s} = \frac{GM}{cr_s}$$

$$\therefore \Delta = \frac{\delta\lambda}{\lambda_E} \sim 2.12 \times 10^{-6}$$

giving red-shift in wave length per wave-length.

errors, (Quinn, earth are not stationary + add error 2nd order quantity
 ② observer Line of sight (1), (2) Can be neglected



(1.36)

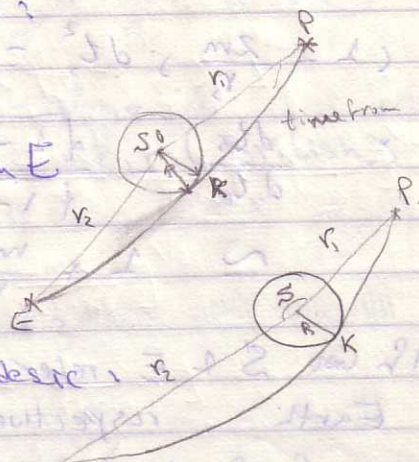
5.6 (also 2.5.5) 2.1 in Phases 7.6 (also 2.5.5)

This result only verified satisfactorily in 1935

4. Radar Echo Delay:

Let RKE be a radar track from the planet P to the earth E grazing the sun's disk at K.

$$OP = r, OE = r_2, OK = R$$



We have seen that for any geodesic, $r^2 \frac{d\phi}{ds} = h$

$$r^2 \frac{d\phi}{ds} = h \quad (1)$$

$$\frac{dt}{ds} = K \left(1 - \frac{2m}{r} \right) \quad (2)$$

$$G = 2\pi h$$

in the Schw. space time

$$ds^2 = \left(1 - \frac{2m}{r} \right) dt^2 - dr^2 \left(1 - \frac{2m}{r} \right) - r^2 (d\theta^2 + 2\sin\theta d\phi^2) \quad (3)$$

From (1), (2)

$$r^2 \frac{d\phi}{dt} = \frac{h}{K} \left(1 - \frac{2m}{r} \right)$$

$$\therefore \frac{d\phi}{dt} = \frac{h}{Kr^2} \left(1 - \frac{2m}{r} \right) \quad (4)$$

But since the radar is null-geodesic, i.e. $ds = 0$; we get from (3)

$$0 = \left(1 - \frac{2m}{r} \right) dt^2 - \left(\frac{dr}{dt} \right)^2 \left(1 - \frac{2m}{r} \right) - r^2 \left(\frac{d\phi}{dt} \right)^2$$

$$\therefore \left(\frac{dr}{dt} \right)^2 = \left(1 - \frac{2m}{r} \right)^2 - \frac{h^2}{K^2 r^2} \left(1 - \frac{2m}{r} \right)^3$$

$$\left[\left(1 - \frac{2m}{r} \right) \left[1 - \frac{h^2}{k^2 r^2} \left(1 - \frac{2m}{r} \right) \right] \right]$$

$$\therefore \frac{dr}{dt} = \pm \left(1 - \frac{2m}{r} \right) \left[1 - \frac{h^2}{k^2 r^2} \left(1 - \frac{2m}{r} \right) \right]^{1/2} \quad (15)$$

To find out the values of the two constants of integration

h, k ; note that

at k_{min} $\frac{dr}{dt} = 0$ when $r = R$ (min)

$$\text{From (5), } \frac{h^2}{k^2} = \frac{R^2}{(1 - 2m/R)} \quad (16)$$

$\therefore$ (5) becomes:

$$\frac{dr}{dt} = \pm \left(1 - \frac{2m}{r} \right) \left[1 - \frac{R^2}{r^2} \left(\frac{1 - 2m/R}{1 - 2m/r} \right) \right]^{1/2}$$

Let time taken from P to E to be given

by

$$T = h(R, r_1) + f(R, r_2)$$

Small quantities $\left\{ \begin{array}{l} \frac{m}{R} \\ \frac{m}{R} \\ \left(\frac{h^2}{R^2} \right) \approx 0 \\ \left(\frac{m^2}{R^2} \right) \approx 0 \end{array} \right.$

$$\text{where } h(R, r_1) = \int_R^{r_1} \frac{dr}{D}$$

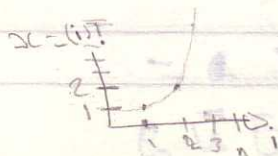
mass of the moon

$$D = \left(1 - \frac{R^2}{r^2} \right)^{1/2} \left[1 - \frac{2m}{r} - \frac{mR}{r(r+R)} \right] \quad (15)$$

$$f(R, r_1) = \int_R^{r_1} \frac{dr}{\left(1 - \frac{2m}{r} \right) D} = \int_R^{r_1} \frac{dr}{D}$$

$$D = \left(1 - \frac{2m}{r} \right) \left[1 - \frac{R^2}{r^2} \left(1 - \frac{2m}{r} \right) \left(1 - \frac{2m}{R} \right) \right]^{1/2}$$

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2} x^2 \approx 1 + nx$$



$$\sum_{n=0}^{\infty} (1+x)^n$$

$x=1$

(138)

$$\text{Therefore, } D = \left(1 - \frac{2m}{r}\right) \left[1 - \frac{R^2}{r^2} \left(1 - \frac{2m}{r}\right) \left(1 + \frac{2m}{r}\right)\right]^{1/2}$$
$$= \left(1 - \frac{2m}{r}\right) \left[1 - \frac{R^2}{r^2} \left(1 - \frac{2m}{r} + \frac{2m}{r}\right)\right]^{1/2}$$

$$D = \left(1 - \frac{2m}{r}\right) \left[1 - \frac{R^2}{r^2} - \frac{2mR^2}{r^2} \left(\frac{1}{R} - \frac{1}{r}\right)\right]^{1/2}$$

$$= \left(1 - \frac{2m}{r}\right) \left(1 - \frac{R^2}{r^2}\right)^{1/2} \left[1 - \frac{2mR^2}{r^2} \cdot \frac{r-R}{rR} \cdot \frac{r^2}{r^2-R^2}\right]^{1/2}$$

$$D = \left(1 - \frac{R^2}{r^2}\right)^{1/2} \cdot \left(1 - \frac{2m}{r}\right) \left[1 - \frac{2mR}{r(r+R)}\right]^{1/2}$$

$$\approx \left(1 - \frac{R^2}{r^2}\right)^{1/2} \cdot \left(1 - \frac{2m}{r}\right) \left[1 - \frac{2mR}{r(r+R)}\right]$$

$$\approx \left(1 - \frac{R^2}{r^2}\right)^{1/2} \left(1 - \frac{2m}{r} - \frac{mR}{r(r+R)}\right)$$

$$\therefore h(R, r) = \int_R^r \left(1 - \frac{R^2}{r^2}\right)^{1/2} \left(1 + \frac{2m}{r} + \frac{mR}{r(r+R)}\right) dr$$

Hence the corresponding in the absence of Sun is obtained by putting $m = 0$ into the above relation.

$$R_0 = \int \left(1 - \frac{R^2}{r^2}\right)^{1/2} dr$$

This shows that there is a time delay which ~~was~~ can be calculated.

The estimated effect in the case of Mercury, was found to be $200 \mu\text{Sec}$.

This result was confirmed in (1975) by observations taken by Space-Crafts Mariner 6 & 7.

Einstein's Field Equations for a Continuous Distribution of Matter.

1. The Material-Energy Tensor:

Consider a Continuous distribution of matter with density ρ , and the velocity of its constituent particles are given by

$$v^\mu = \frac{dx^\mu}{dt} \quad \mu = 1, 2, 3 \rightarrow 0$$

All the Classical properties of such a distribution can be described by the tensor

$$T^{\mu\nu} = \rho \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \quad \mu, \nu = 1, 2, 3, 4$$

$$T^{\mu\nu} = \begin{bmatrix} \rho u^2 & \rho uv & \rho uw & \rho u \\ \rho uv & \rho v^2 & \rho vw & \rho v \\ \rho uw & \rho vw & \rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix} \quad (3)$$

If this distribution is atomically constituted, we can study its mechanics (behaviour) by studying the motion of its C.G. ($\bar{u}, \bar{v}, \bar{w}$) and the motion of its particles relative to C.G. (u', v', w')

$u = \bar{u} + u' \quad , \quad v = \bar{v} + v'$

$\underline{v_p} = \underline{v_g} + \underline{v_{gp}}$

Hence the terms in (3) will give rise to a Stress term

$$\begin{aligned} \text{e.g. } \sum \rho uv &= \sum \rho (\bar{u} + u')(\bar{v} + v') \\ &= \sum \rho \bar{u}\bar{v} + \bar{v} \sum \rho u' + \bar{u} \sum \rho v' + \sum \rho u'v' \\ &= \rho \bar{u}\bar{v} + 0 + 0 + P_{xy} \end{aligned}$$

(140)

$\Rightarrow$ $\rho \bar{u} \bar{v}$...

In this way the tensor (3) will take the form

$$T^{\mu\nu} = \begin{bmatrix} \rho u^2 + p_{xx} & \rho uv + p_{xy} & \rho uw + p_{xz} & \rho u \\ \rho v u + p_{yx} & \rho v^2 + p_{yy} & \rho vw + p_{yz} & \rho v \\ \rho w u + p_{zx} & \rho w v + p_{zy} & \rho w^2 + p_{zz} & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix}$$

where (u, v, w) here refers to the velocity of

C.G. i.e. $(\bar{u}, \bar{v}, \bar{w})$

Elasticity, ... etc.

Then the laws of Classical hydrodynamics
can be summed into the relation given by

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \quad \mu = 1, 2, 3, 4 \quad (5)$$

For example, taking $\mu = 4$:

$$\frac{\partial T^{41}}{\partial x^1} + \frac{\partial T^{42}}{\partial x^2} + \frac{\partial T^{43}}{\partial x^3} + \frac{\partial T^{44}}{\partial x^4} = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Which is the classical eqnⁿ of Continuity

$$\text{cont. } \nabla \cdot (\rho \underline{v}) = - \frac{\partial \rho}{\partial t}$$

Geometry represents physics but the p's for Varied

Einstein's Geometers

he found this Sake

geometry
geometrization

versus

quantization

2. The Four Identities - Bianchi's Identity:

Profile of tensors have special coordinates - Geodesic Coordinates
utilize in the form of tensors $\rightarrow$ correct all the way

Note: In any Riemannian space, it is always possible to use "geodesic Coordinates": at the origin of which the 1st partial derivatives of the metric tensor vanish, while its second and higher derivatives "Do not vanish"

$$\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right]_{\text{Origin}} = 0 \quad \text{"in geodesic coordinates"}$$

Thus using Geodesic Coordinates in the Riemannian Space, the Riemann-Christoffel tensor will take the form (at the origin of geodesic Coordinates)

$$B_{\mu\nu\sigma}^{\epsilon} = \{ \epsilon_{\mu\sigma} \}_{,\nu} - \{ \epsilon_{\mu\nu} \}_{,\sigma} + \{ \alpha_{\mu\sigma} \} \{ \epsilon_{\alpha\nu} \} + \{ \alpha_{\mu\nu} \} \{ \epsilon_{\alpha\sigma} \}$$

$$(B_{\mu\nu\sigma}^{\epsilon})_O = \{ \epsilon_{\mu\sigma} \}_{,\nu} - \{ \epsilon_{\mu\nu} \}_{,\sigma} + 0 + 0$$

"at Origin in geodesic Coordinates"

Hence, we can easily prove that, in geodesic Coordinates at O , Covariant differentiation will reduce to ordinary partial differ.

$$B_{\mu\rho\sigma\epsilon}^{\epsilon} + B_{\mu\sigma\epsilon\nu}^{\epsilon} + B_{\mu\epsilon\nu\sigma}^{\epsilon} = 0 \quad \text{"Verify"}$$

"at O only"

(142)

* (ask)

∴ This relation is true everywhere in geodesic Coordinates, since it is a Covariant Form.

Since it is true in geodesic Coordinates, it will be true in any other Coordinate System i.e. it is True Always.

It is known as the Bianchi's Identity

Cycle



$$B_{\mu\nu\sigma}^{\epsilon} + B_{\mu\sigma\nu}^{\epsilon} + B_{\mu\nu\sigma}^{\epsilon} = 0$$

Contracting this identity by setting $\sigma = \epsilon$,

we get

Remember the antisymmetries while substituting signs

$$R_{\mu\nu\epsilon}^{\epsilon} - R_{\mu\epsilon\nu}^{\epsilon} + g^{\epsilon\alpha} B_{\alpha\mu\nu\epsilon} = 0$$

(Identically zero)

$$\therefore R_{\mu\nu\epsilon}^{\epsilon} - R_{\mu\epsilon\nu}^{\epsilon} + g^{\epsilon\alpha} B_{\epsilon\mu\nu\alpha} = 0$$

No change

$$(g^{\mu\alpha} R_{\mu\nu\epsilon}^{\epsilon} - g^{\mu\alpha} R_{\mu\epsilon\nu}^{\epsilon} + g^{\mu\alpha} g^{\epsilon\alpha} B_{\epsilon\mu\nu\alpha}) = 0$$

Invariant

$$R_{\nu\epsilon}^{\epsilon} - R_{\epsilon\nu}^{\epsilon} + g^{\epsilon\alpha} R_{\nu\alpha\epsilon}^{\epsilon} = 0$$

$$R = g^{\mu\alpha} R_{\mu\alpha}^{\mu} = R_{\mu}^{\mu}$$

$$R_{\nu\epsilon}^{\epsilon} - R_{\epsilon\nu}^{\epsilon} + R_{\nu\epsilon}^{\epsilon} = 0$$

$$R_{\nu\epsilon}^{\epsilon} - R_{\epsilon\nu}^{\epsilon} + R_{\nu\epsilon}^{\epsilon} = 0$$

"also 0 to"

$$\left. \begin{aligned} R_{11}^1 + R_{22}^2 + R_{33}^3 + R_{44}^4 - \frac{1}{2} R_{11} &= 0 & R_{21}^1 + R_{32}^2 + R_{43}^3 + R_{14}^4 - \frac{1}{2} R_{22} &= 0 \\ R_{11}^1 + R_{22}^2 + R_{33}^3 + R_{44}^4 - \frac{1}{2} R_{22} &= 0 & R_{11}^1 + R_{22}^2 + R_{33}^3 + R_{44}^4 - \frac{1}{2} R_{33} &= 0 \end{aligned} \right\} \text{ (143)}$$

When $\epsilon = \nu \Rightarrow (R_{\mu\nu})_{;\epsilon}$ becomes $R_{\nu;\nu} = R_{;\nu} \Rightarrow R_{;\nu} = 0$

$$\therefore R_{\nu;\epsilon} - \frac{1}{2} R_{;\nu} = 0 \quad \text{[1]} \quad \nu = 1, 2, 3, 4$$

[1st Running Variable is (1), 2nd summation over (1)]

This relation is known as the 4-Identities

$$\nu = 1, 2, 3, 4 \Rightarrow R_{\mu;\nu}^{\nu}$$

$$\therefore \left(R_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} R \right)_{;\nu} = 0 \quad \text{[1]}$$

when $\mu = \nu \Rightarrow R_{\mu}^{\mu} = R$ and $\Rightarrow$ L.H.S. $= \frac{1}{2} R_{;\nu} = 0$

$\Rightarrow$ in both quantities of L.H.S. the running variable is (μ) and there is summation over (ν)

This shows that the Covariant divergence of

$$\left[\text{let } R_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} R = G_{\mu}^{\nu} \Rightarrow \partial_{\mu\nu} (R_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} R) = \partial_{\mu\nu} G_{\mu}^{\nu} = G_{\mu\nu} = 0 \right]$$

$$\text{the tensor } \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = G_{\mu\nu} \quad \text{[2]}$$

Of course [2] is found from [1]

vanishes identically

after removing the Covariant derivative w.r to (ν).

Which is Einstein's Field Equations for a

Material Distribution is taken to be

$$[\partial^{\mu\nu} \text{ [2]} = \text{[3]} = \partial^{\mu\nu} G_{\mu\nu} = G_{\mu}^{\mu} = -\kappa T_{\mu}^{\mu}]$$

$$R_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} R = -\kappa T_{\mu}^{\nu} \quad \text{[3]}$$

$$(g^{\mu\mu} \text{ [3]})_{;\nu} \text{ gives } (T^{\mu\mu})_{;\nu} = 0 \quad \left[\begin{array}{l} \text{Is it still correct at} \\ \text{the origin only?} \end{array} \right]$$

(κ) is a Constant to be determined

$$R_{11}^1 + R_{22}^2 + R_{33}^3 + R_{44}^4 - \frac{1}{2} R (g_1^1 + g_2^2 + g_3^3 + g_4^4) = 0 \quad (\mu = 1, \nu = 1, 2, 3, 4) \quad \text{[1]}$$

and for this reason [1] = [1]

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Notes:

Classical Idea

Material - energy tensor

$$T^{\mu\nu} = \rho \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \quad (1)$$

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0 \quad (2)$$

1st Einstein generalized these relations to be into the form.

He defined the material - energy tensor by ρ_0 Proper density = ρ .

$$T^{\mu\nu} = \rho_0 \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \quad (1')$$

where ρ_0 is the Proper density.

$$(T^{\mu\nu})_{;\nu} = 0 \quad (2')$$

$$(T^{\mu\nu})_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \{\mu\alpha\nu\} T^{\alpha\mu} + \{\nu\alpha\mu\} T^{\mu\alpha} = 0$$

2nd To get relation between ρ_0 & ρ note that from (1)

$$T^{\mu\nu} = \rho \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \left(\frac{ds}{dt} \right)^2 \quad c^2 =$$

without c $\frac{ds}{dt} = 1$ Stationary in space only

$$= c^2 \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \quad (3)$$

From (1') & (3) we get $\rho_0 = c^2 \rho$ (classical) (4)

Knowing that the Covariant divergence of a certain tensor in the Riemannian Space vanishes identically

$$(R^\nu_\mu - \frac{1}{2} g^\nu_\mu R)_{;\nu} = 0 \quad (4 - \text{Identities})$$

Then Einstein was lead to choose his general field equations for any material distribution to be

$$T^{\mu\nu} = \rho_0 \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \quad \text{We'll outside}$$

$$R^\nu_\mu - \frac{1}{2} g^\nu_\mu R = -k T^\nu_\mu \quad \text{We'll inside}$$

3rd

* To get the field eqns for empty space from (iii)

we put $T^{\mu\nu} = 0$ in (iii) (14)

From (iii) $R^\nu_\mu - \frac{1}{2} g^\nu_\mu R = 0$

$$\delta^\mu_\mu = \delta^\mu_\mu = 4$$

Contracting : $R - \frac{1}{2} \times 4 R = 0$

$$\therefore R - 2R = 0$$

$$R = 0$$

Hence we get $R^\nu_\mu = 0$

or $R_{\mu\nu} = 0$

(I)

Note also, if we Contract (iii) directly we get

$$R - \frac{1}{2} \times 4 R = -k T$$

$$R - 2R = -k T$$

$$\Rightarrow R = k T$$

(ii) is Sometimes put into the form

$$R^\nu_\mu = -k \left(T^\nu_\mu - \frac{1}{2} g^\nu_\mu T \right)$$

$$R_{\mu\nu} = -k \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

To get k , and Test according to Classical Mechanics.

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4th To get Poisson's Equations as a First eq approximation from Einstein's Field Eqns - (Value of κ)

To Consider a very weak, Static, field whose metric differs slightly from that of the flat space of Special Relativity (The Galilean Frame)

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad (1)$$
$$= ds^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (2)$$

Taking the particle's of the fluid to be nearly at rest

i.e $\frac{dx}{ds} \approx 0, \quad \frac{dy}{ds} \approx 0, \quad \frac{dz}{ds} \approx 0 \quad (3)$

From (1), we set $\left(\frac{ds}{dt}\right)^2 = c^2$

$$\therefore \frac{ds}{dt} = c$$

$$g = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -1$$

Also from (2) $g_{11} \approx -1, \quad g_{22} \approx -1, \quad g_{33} \approx -1$

$$g_{44} \approx 1, \quad g_{\mu\nu} \approx 0 \quad \text{if } \mu \neq \nu$$

Substi. in (II), we get the non-vanishing components of the material-energy tensor

$$T^{\mu\nu} = \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

$$T^{44} = T_4^4 = T = \rho$$

$$\begin{aligned} T_4^4 &= g_{4\alpha} T^{\alpha 4} \\ g_{44} T^{44} &= 1 \times \rho \\ &= \rho \end{aligned}$$

(Pen and Paper and think) always

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From (II), (iii), we get

$$R_{44} = g - k \left(T_{44} - \frac{1}{2} g_{44} T \right) = -k \left(c_0 - \frac{1}{2} \times 1 \times c_0 \right) \Rightarrow R_{44} = -\frac{1}{2} k c_0 \quad (4)$$

But $R_{44} = \frac{\partial^2 \log \sqrt{-g}}{\partial x^\alpha \partial x^\alpha} - \left\{ \begin{matrix} \alpha \\ 44 \end{matrix} \right\} \frac{\partial \log \sqrt{-g}}{\partial x^\alpha}$

$g = -1$

$$= \left\{ \begin{matrix} \alpha \\ 44 \end{matrix} \right\}_{,\alpha} + \left\{ \begin{matrix} \alpha \\ 4\alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ 4\beta \end{matrix} \right\} = 0$$

0 = 2nd order = $\left\{ \begin{matrix} \alpha \\ 44 \end{matrix} \right\}_{,\alpha} + 2nd \text{ order terms}$

$$\therefore R_{44} = - \left\{ \begin{matrix} \alpha \\ 44 \end{matrix} \right\}_{,\alpha} \quad (5)$$

Since from previously $\left\{ \begin{matrix} \alpha \\ 44 \end{matrix} \right\}_{,\alpha} = -\frac{1}{2} g^{\alpha\alpha} \frac{\partial g_{44}}{\partial x^\alpha} = 1$

$$= -\frac{1}{2} (-1) \frac{\partial g_{44}}{\partial x^\alpha} \quad \text{for } \alpha = 1, 2, 3$$

$$= \frac{1}{2} \frac{\partial g_{44}}{\partial x^\alpha} \quad \alpha = 1, 2, 3 \quad (6)$$

$$\therefore R_{44} = -\frac{1}{2} \left[\frac{\partial^2 g_{44}}{\partial x^2} + \frac{\partial^2 g_{44}}{\partial y^2} + \frac{\partial^2 g_{44}}{\partial z^2} \right] = -\frac{1}{2} \nabla^2 g_{44} \quad (7)$$

Taking $\psi = \frac{c^2 (g_{44} - 1)}{2}$ "as before"

$$\therefore \nabla^2 g_{44} = \frac{2}{c^2} \nabla^2 \psi \quad (8)$$

From (7), (8)

$$\therefore R_{44} = -\frac{1}{c^2} \nabla^2 \psi \quad (9)$$

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From (4) and (9)

$$R_{44} = -\frac{1}{c^2} \nabla^2 \Psi = -\frac{1}{2} K \rho$$

$$\therefore \frac{1}{c^2} \nabla^2 \Psi = \frac{1}{2} K \rho$$

$$\therefore \nabla^2 \Psi = \frac{c^2}{2} K \rho$$

Which is The General Relativistic form.

But in Classical mechanics

$$\nabla^2 \Psi = 4\pi \gamma \rho \quad (11)$$

If we have to get (11), an identical

$$\therefore \frac{1}{2} c^2 K \rho = 4\pi \gamma \rho$$

Hence the relativistic Constant K will be given by

$$K = \frac{8\pi \gamma}{c^2} \approx 2.073 \times 10^{-48} \text{ cm}^{-1} \cdot \text{gm}^{-1} \cdot \text{sec}^2$$

$$\boxed{\text{Force} = \text{gm} \cdot \frac{\text{cm}}{\text{sec}^2} = \frac{\text{gm} \cdot \text{cm}}{\text{sec}^2}}$$

In Relativistic-Heavy Side units (in Cosmology)

$$c=1, \quad \gamma=1, \quad K=8\pi$$

In these units Einstein's Field eq^{ns} will take

$$\text{The form } R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}$$

Interior

Schwarzschild's Exterior Solution:

Consider a sphere of homogeneous, perfect fluid at rest at the origin of coordinates

$$\rho_0 = \text{const}$$

The metric of the Riemannian space representing this material distribution can simply be taken into the form

$$ds^2 = e^{2\lambda} dt^2 - e^{2\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

$$\lambda = \lambda(r), \quad \lambda = \lambda(r)$$

The Material-energy tensor for this distribution was taken to be

$$T_1^1 = T_2^2 = T_3^3 = -P_0 \quad (\text{Isotropic Pressure})$$

$$T_4^4 = \rho_0$$

Then substitute into Einstein's Field eqns

$$R_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} R = -8\pi T_{\mu}^{\nu} \quad (2)$$

We get

1st Christoffel

2nd Ricci tensor

$$R_{\mu}^{\nu} = R_1^1, R_2^2, R_3^3, R_4^4$$

$$R_{\epsilon}^{\epsilon} = R = R_1^1 + R_2^2 + R_3^3 + R_4^4$$

Then in (2)

(Home work)

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we get

From T_1 , $8\pi P_0 = e^{-2\lambda} \left(\frac{2\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}$ (i)

From T_2^2, T_3^3 , $8\pi P_0 = e^{-2\lambda} (\nu'' - \lambda'\nu' + \nu'^2 + \frac{\nu'}{r} - \frac{\lambda'}{r})$ (ii)

From T_4^4 , $8\pi \rho_0 = -e^{-2\lambda} \left(\frac{2\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}$ (iii)

(Classically) $\frac{dP}{dr} + \rho \frac{d\psi}{dr} = 0$ (*)

From (i), (ii), (iii) we get

$\frac{dP_0}{dr} + \nu' (P_0 + \rho_0) = 0$ (iv)



① Get (i) = (iii)

② Get $\frac{dP_0}{dr}$ from (iv) and use the result

③ we form $(P_0 + \rho_0)$ i.e. (i) + (iii)

we get (iv) the analogue of (*)

$R_{33} = \sin^2\theta R_{22}$
 $R_3^3 = R_2^2$

Solution: From (i), $\times r^2$

$e^{-2\lambda} (-2\lambda' r + 1) = 1 - 8\pi \rho_0 r^2$

$\frac{d(r e^{-2\lambda})}{dr} = 1 - 8\pi \rho_0 r^2$

$\therefore r e^{-2\lambda} = r - \frac{8\pi \rho_0}{3} r^3 + \text{Const.}$

$e^{-2\lambda} = 1 - \frac{r^2}{R^2} + \frac{C}{r}$ (v)

to avoid at $r=0 \rightarrow$ gives ∞

To avoid singularities at the origin we take $C=0$

$$e^{-2\lambda} = 1 - \frac{r^2}{R^2} \quad (vi)$$

where $\frac{1}{R^2} = \frac{8}{3} \pi \epsilon_0$

From (vii) we have

$$\frac{dP_0}{dr} + \nu'(P_0 + P_0) = 0$$

$$\int \frac{dP_0}{P_0 + P_0} = - \int \nu' dr \quad \frac{P_0 = \text{const.}}{15}$$

$$\log(P_0 + P_0) = -\nu + \text{const.}$$

$$P_0 + P_0 = K e^{\nu}$$

From (i), (vii) we get $8\pi(P_0 + P_0) = e^{-2\lambda} \left[\frac{2}{r} (\nu' + \lambda') \right]$

$$\therefore 8\pi(P_0 + P_0) = 2 \frac{e^{-2\lambda}}{r} (\nu' + \lambda')$$

$$K e^{\nu} = P_0 + P_0 = \frac{1}{4\pi} \cdot \frac{e^{-2\lambda}}{r} (\lambda' + \nu')$$

$$e^{(\nu-2\lambda)} (\lambda' + \nu') = C r \quad C = \text{const.} / r \quad (16)$$

$$e^{3\lambda} (e^{(\nu-2\lambda)} (d\lambda + d\nu)) = e^{3\lambda} C r dr \quad (17)$$

$$\int e^{(\nu+1\lambda)} (d\lambda + d\nu) = \int \frac{C r}{(1 - r^2/R^2)^{3/2}} dr$$

Integrating w.r to $r \Rightarrow$

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$$\int e^{(1+\lambda)} d(1+\lambda) = \int \frac{r}{(1 - (r/R)^2)^{3/2}} dr = \int \frac{r}{(1 - (r/R)^2)^{3/2}} dr$$

$$= A \frac{1}{\sqrt{1 - (r/R)^2}} + \text{Const.}$$

$$e^{\nu} = A + B \sqrt{1 - (r/R)^2}$$

(Square root of $1 - (r/R)^2$)

Hence we get the solution

$$ds^2 = (A - B \sqrt{1 - (r/R)^2})^2 dt^2 - \frac{dr^2}{(1 - (r/R)^2)^{1/2}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where $\frac{1}{R^2} = \frac{8\pi}{3} \rho_0$

Still we want A, B and this solutions must satisfy

A, B are two constants of integration to be determined.

* Boundary Conditions:

(i) The exterior solution

$$ds^2 = -dt^2 - \frac{dr^2}{1 - 2M/r} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2)$$

and the interior solution (1) should be identical

when $r = a$

Bianchi's four identities :

$$\star \quad B_{\mu\nu\sigma\rho}^{\epsilon} + B_{\rho\sigma\epsilon\mu}^{\epsilon} + B_{\mu\epsilon\rho\nu}^{\epsilon} = 0$$

At the origin of geodesic coordinates we have :

$$B_{\mu\nu\sigma}^{\epsilon} = \{^{\epsilon}_{\mu\sigma}\}_{,\nu} - \{^{\epsilon}_{\mu\nu}\}_{,\sigma} + 0 - 0 \Rightarrow$$

$$\{^{\epsilon}_{\mu\sigma}\}_{,\nu} = \frac{1}{2} \frac{\partial}{\partial x^{\nu}} \left[g^{\epsilon\alpha} \left[\frac{\partial g_{\mu\alpha}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\alpha}} \right] \right] = \frac{1}{2} g^{\epsilon\alpha} \frac{\partial}{\partial x^{\nu}} \left[\frac{\partial g_{\mu\alpha}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\alpha}} \right]$$

$$= g^{\epsilon\alpha} \frac{\partial}{\partial x^{\nu}} ([\mu\sigma, \alpha]) \quad \text{Also } \{^{\epsilon}_{\mu\nu}\}_{,\sigma} = g^{\epsilon\alpha} \frac{\partial}{\partial x^{\sigma}} ([\mu\nu, \alpha])$$

$$\Rightarrow B_{\mu\nu\sigma}^{\epsilon} = g^{\epsilon\alpha} \left[\frac{\partial}{\partial x^{\nu}} ([\mu\sigma, \alpha]) - \frac{\partial}{\partial x^{\sigma}} ([\mu\nu, \alpha]) \right] \Rightarrow$$

$$B_{\mu\nu\sigma\rho}^{\epsilon} = \frac{1}{2} g^{\epsilon\alpha} \left[\frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\sigma}} + \frac{\partial^2 g_{\sigma\alpha}}{\partial x^{\nu} \partial x^{\rho}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu} \partial x^{\alpha}} - \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\rho} \partial x^{\sigma}} - \frac{\partial^2 g_{\rho\alpha}}{\partial x^{\nu} \partial x^{\mu}} + \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\rho} \partial x^{\mu}} \right]$$

$$\Rightarrow B_{\mu\nu\sigma\rho}^{\epsilon} = g^{\epsilon\alpha} \left[\frac{\partial^2}{\partial x^{\nu} \partial x^{\sigma}} ([\mu\sigma, \alpha]) - \frac{\partial^2}{\partial x^{\nu} \partial x^{\rho}} ([\mu\nu, \alpha]) \right]$$

$$+ \frac{\partial g^{\epsilon\alpha}}{\partial x^{\rho}} \left[\frac{\partial}{\partial x^{\nu}} ([\mu\sigma, \alpha]) - \frac{\partial}{\partial x^{\sigma}} ([\mu\nu, \alpha]) \right]$$

$$\Rightarrow B_{\mu\nu\sigma\rho}^{\epsilon} = g^{\epsilon\alpha} \left[\frac{\partial^2}{\partial x^{\nu} \partial x^{\sigma}} ([\mu\sigma, \alpha]) - \frac{\partial^2}{\partial x^{\nu} \partial x^{\rho}} ([\mu\nu, \alpha]) \right]$$

$$+ \frac{\partial g^{\epsilon\alpha}}{\partial x^{\rho}} \left[\frac{\partial}{\partial x^{\nu}} ([\mu\sigma, \alpha]) - \frac{\partial}{\partial x^{\sigma}} ([\mu\nu, \alpha]) \right]$$

$$\Rightarrow B_{\mu\nu\sigma\rho}^{\epsilon} = g^{\epsilon\alpha} \left[\frac{\partial^2}{\partial x^{\nu} \partial x^{\sigma}} ([\mu\nu, \alpha]) - \frac{\partial^2}{\partial x^{\nu} \partial x^{\rho}} ([\mu\sigma, \alpha]) \right]$$

$$+ \frac{\partial g^{\epsilon\alpha}}{\partial x^{\sigma}} \left[\frac{\partial}{\partial x^{\nu}} ([\mu\nu, \alpha]) - \frac{\partial}{\partial x^{\rho}} ([\mu\sigma, \alpha]) \right]$$

$$\Rightarrow (1) + (2) + (3) =$$

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Contracting

$$R_{\mu\nu\epsilon} - R_{\mu\epsilon\nu} + B_{\mu\nu\epsilon} = 0$$

$$g^{\mu\epsilon} R_{\mu\nu\epsilon} - R_{\mu\epsilon\nu} + B_{\mu\nu\epsilon} = 0$$

$\mu = \nu$
 $\downarrow$
 $g^{\mu\mu}$

$$g^{\mu\mu} R_{\mu\mu\epsilon} - (g^{\mu\epsilon} R_{\mu\epsilon\nu}) + g^{\mu\epsilon} g^{\nu\epsilon} B_{\mu\nu\epsilon} = 0$$

(Symmetric)

$$R_{\nu\epsilon} - R_{\epsilon\nu} + g^{\mu\epsilon} g^{\nu\epsilon} B_{\epsilon\mu\nu\epsilon} = 0$$

$$R_{\nu\epsilon} - R_{\epsilon\nu} + g^{\nu\epsilon} B_{\nu\mu\mu\epsilon}$$

$$g^{\nu\epsilon} R_{\nu\mu\mu\epsilon}$$

$$R_{\nu\epsilon} - R_{\epsilon\nu} + R_{\nu\epsilon} = 0$$

$$2R_{\mu\epsilon\nu} - R_{\epsilon\mu} = 0 \Rightarrow R_{\mu\epsilon\nu} = \frac{1}{2} \delta_{\mu}^{\nu} R_{\epsilon\epsilon}$$

$$R_{\mu\epsilon\nu} = \frac{1}{2} \frac{\partial R}{\partial x^{\mu}} \quad \mu = 1, 2, 3, 4$$

$$(R_{\mu}^{\nu} - \frac{1}{2} g_{\mu}^{\nu} R)_{\epsilon\epsilon} = 0$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

whose divergence is zero.

$$B_{\mu\nu\sigma}^{\epsilon} = \{\mu\sigma\}_{,\nu}^{\epsilon} - \{\mu\nu\}_{,\sigma}^{\epsilon} + \{\mu\sigma\}^{\epsilon}\{\alpha\nu\} - \{\mu\nu\}^{\epsilon}\{\alpha\sigma\}$$

Using geodesic coords

At "0":

$$(i) B_{\mu\nu\sigma}^{\epsilon} = \{\mu\sigma\}_{,\nu}^{\epsilon} - \{\mu\nu\}_{,\sigma}^{\epsilon}$$

(ii) cov diff $\equiv$ ordinary diff

$$B_{\mu\nu\sigma;\rho}^{\epsilon} + B_{\mu\sigma\rho;\nu}^{\epsilon} + B_{\mu\rho\nu;\sigma}^{\epsilon} + B_{\mu\nu\sigma;\rho}^{\epsilon} = 0$$

$$\cancel{B_{\mu\nu\sigma;\rho}^{\epsilon} = \{\mu\sigma\}_{,\nu\rho}^{\epsilon} - \{\mu\nu\}_{,\sigma\rho}^{\epsilon}}$$

$$\cancel{B_{\mu\sigma\rho;\nu}^{\epsilon} = \{\mu\rho\}_{,\sigma\nu}^{\epsilon} - \{\mu\sigma\}_{,\rho\nu}^{\epsilon}}$$

$$\cancel{B_{\mu\rho\nu;\sigma}^{\epsilon} = \{\mu\nu\}_{,\rho\sigma}^{\epsilon} - \{\mu\rho\}_{,\sigma\nu}^{\epsilon}}$$

$$\cancel{B_{\mu\nu\sigma;\rho}^{\epsilon} = \{\mu\sigma\}_{,\rho\nu}^{\epsilon} - \{\mu\nu\}_{,\sigma\rho}^{\epsilon}}$$

$$B_{\mu\nu\sigma;\rho}^{\epsilon} = \{\mu\sigma\}_{,\rho\nu}^{\epsilon} - \{\mu\nu\}_{,\sigma\rho}^{\epsilon}$$

للاستاذ الدكتور

شكر ونكري



فهمي

ابراهيم

ميخائيل

تقديم الاسرة

بالشكر

G. Rel. Boundary Conditions

 $\rho_0 = \text{const}$

$$A - B\sqrt{1 - \frac{a^2}{R^2}} = \sqrt{1 - \frac{2m}{a}} \quad (3)$$



$$1 - \frac{a^2}{R^2} = 1 - \frac{2m}{a}$$

(4) m is a representative of mass

$$\frac{2m}{a} = \frac{a^2}{R^2} = \frac{a^2}{1} \times \frac{8}{3} \pi \rho_0$$

Pressure = stress decreases till we reach the surface

$$m = \frac{4}{3} \pi a^3 \rho_0$$

Showing that the Constant m appearing in the ~~Schwarzschild~~~~Schwarzschild's~~ exterior solution corresponds to the

Same sphere assumed in the interior solution

(2) The stress P_0 should vanish at the boundaryi.e. $P_0 = 0$ where $r = a$

But from (i),

$$8\pi P_0 = \frac{1}{R^2} \left(\frac{3B\sqrt{1 - (r/R)^2} - A}{A - B\sqrt{1 - (r/R)^2}} \right)$$

$$3B\sqrt{1 - \frac{a^2}{R^2}} - A = 0 \quad (6)$$

$$(3) + (6) \rightarrow 2B\sqrt{1 - \frac{a^2}{R^2}} = \sqrt{1 - \frac{a^2}{R^2}}$$

$$\therefore B = \frac{1}{2}$$

from (6) $\Rightarrow$

$$A \approx \frac{3}{2} \sqrt{1 - \frac{a^2 \cos^2 \theta}{R^2}} \quad \theta = \frac{\pi}{6} \Rightarrow 1/8 = A$$

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$R_{11} = \nu'' + \nu'^2 - \lambda' \nu' - \frac{2\lambda'}{r}$$

$$R'_i = g'' R_{ii} = -e^{-2\lambda} (v'' + v'^2 - \lambda' v' - \frac{2\lambda'}{r})$$

$$R_{22} = e^{-2\lambda} [r(\psi' - \lambda') + 1] - 1$$

$$R_2^2 = -e^{-2\lambda} \left[\frac{2r - \lambda}{r} + \frac{1}{r^2} \right] + \frac{1}{r^2}$$

$R \cap S = \{x \mid x \text{ is a number and } x \text{ is a letter}\}$

Invariant Form:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2$$

$$= dt^2 \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 - c^2 \right]$$

$$= dt'^2 \left[\left(\frac{dx'}{dt'} \right)^2 + \left(\frac{dy'}{dt'} \right)^2 + \left(\frac{dz'}{dt'} \right)^2 - c^2 \right]$$

$$\therefore [U^2 - c^2] = [U'^2 - c^2] \left(\frac{dt'}{dt} \right)^2$$

$$(1 - U^2/c^2) = (1 - U'^2/c^2) \left(\frac{dt'}{dt} \right)^2$$

Using inverse Lorentz transfo.

$$dt = \beta \left(dt' + \frac{v}{c^2} dx' \right)$$

$$\frac{dt}{dt'} = \beta \left(1 + \frac{vu'}{c^2} \right) = \frac{(1 - vu'/c^2)}{\sqrt{1 - (v/c)^2}}$$

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$$B_{\mu\nu\sigma\epsilon}^{\epsilon} = \cancel{\left\{ \begin{matrix} \epsilon \\ \mu\sigma \end{matrix} \right\}_{\nu\epsilon}} - \cancel{\left\{ \begin{matrix} \epsilon \\ \mu\nu \end{matrix} \right\}_{\sigma\epsilon}} \quad (1)$$

$$B_{\mu\sigma\epsilon\nu}^{\epsilon} = \cancel{\left\{ \begin{matrix} \epsilon \\ \mu\epsilon \end{matrix} \right\}_{\sigma\nu}} - \cancel{\left\{ \begin{matrix} \epsilon \\ \mu\sigma \end{matrix} \right\}_{\epsilon\nu}} \quad (2)$$

$$B_{\mu\epsilon\nu\sigma}^{\epsilon} = \cancel{\left\{ \begin{matrix} \epsilon \\ \mu\nu \end{matrix} \right\}_{\epsilon\sigma}} - \left\{ \begin{matrix} \epsilon \\ \mu\epsilon \end{matrix} \right\}_{\nu\sigma} \quad (3)$$

$$(1) + (2) + (3) = 0$$